Part I
Fluid Flow

1 Thermodynamics

A thermodynamic system consists of a body of matter isolated from its surroundings. We require two independent intensive thermodynamic variables to define the state of matter inside a system.

- Intensive variables: pressure, temperature, density
- Extensive variables: volume, mass
- Equation of state: connects variables. example $p = p(\rho, T)$
- Process: takes you from state 1 to state 2 for a path parametrized by Temperature and pressure, for example

We write changes in state in terms of state variables,

$$\partial p = \left(\frac{\partial p}{\partial \rho}\right)_T + \left(\frac{\partial p}{\partial T}\right)_\rho$$

1.0.1 Extensive Variables

- System mass M
- Total volume V
- Total Energy E
- Total Enthalpy: a thermodynamic quantity equivalent to the total heat content of a system.

$$H = H(\rho, T) = E + pV$$

- Total Kinetic Energy K
- Heat Q
- Work done on a system W

1.0.2 Intensive Variables

- Specific volume $V = \frac{V}{M} = \frac{1}{\rho}$
- Internal Energy: the energy contained within the system, excluding the kinetic energy of motion of the system as a whole and the potential energy of the system as a whole due to external force fields. It keeps account of the gains and losses of energy of the system that are due to changes in its internal state.

$$e = e(\rho, T) = \frac{E}{M}$$

  This is the variable of choice for closed systems

- Enthalpy: a thermodynamic quantity equivalent to the total heat content of a system.

$$h = h(\rho, T) = e + pv$$

  This is the variable of choice for an open system

- Entropy: a thermodynamic quantity representing the unavailability of a system's thermal energy for conversion into mechanical work, often interpreted as the degree of disorder or randomness in the system.

$$s = s(\rho, T) = \frac{S}{M}$$

1.0.3 Gas Models

- Real Gas $e = e(\rho, T)$, $c_v = c_v(\rho, T)$
- Ideal Gas $e = e(\rho, T)$, $c_v = c_v(\rho, T)$ or $e = e(T)$, $c_v = c_v(T)$, depending on the gas (obeys the ideal gas law)
- Thermally Perfect Gas $e = e(T)$, $c_v = c_v(T)$
- Calorically Perfect Gas $e = e(T)$, $c_v = const$

Equations of State

- Ideal Gas

$$p = \frac{RT}{v} = \rho RT$$

- Clausius 1

$$p = \frac{RT}{v - b} \qquad \text{where b is a fitting parameter}$$

- Van der Waals

$$p = \frac{RT}{v - b} - \frac{a}{v^2} \qquad \text{where a,b are fitting parameters}$$

1.0.4 Specific Heat

Specific heat is the heat needed to raise the temperature of a unit mass of gas by 1 degree. This is process-dependent, as this occurs at constant pressure or constant volume.

$$C = \frac{\partial Q}{\partial T} \text{ (extensive)} \qquad c = \frac{\partial q}{\partial T} \text{ (intensive)}$$

$$c_v = \left(\frac{\partial q}{\partial T}\right)_v \qquad c_p = \left(\frac{\partial q}{\partial T}\right)_p$$

1.0.5 Adiabatic index

The adiabatic index is the heat capacity ratio

$$\gamma = \frac{c_p}{c_v} = \gamma(\rho, T)$$

Note that for an ideal gas,

$$c_v = \frac{R}{\gamma - 1} \qquad c_p = \frac{\gamma R}{\gamma - 1}$$

Kinetic Theory predicts that

- For monotomic molecules, $\gamma = \frac{5}{3}$
- Diatomic molecules (such as air) $\gamma = \frac{7}{5}$
- Polyatomic molecules $\gamma = \frac{8}{6}$

In general, $e = e(\rho, T)$ and $h = h(\rho, T)$ and we require an equation of state. Recall that for a perfect gas,

$$e(T) = \int c_v dT = c_v T + e_0$$

If one assumes $e_0 = 0$, then $e = c_v T$ and $h = c_p T$ so then $\gamma = const.$
We can develop a caloric equation of state

$$e = e(p, v) = c_v T = \frac{RT}{\gamma - 1} = \frac{pv}{\gamma - 1} = \frac{p}{\rho}\frac{1}{\gamma - 1} \implies p = (\gamma - 1)\rho e$$

The precludes the usage of temperature. which is often not necessary in compressible non-reacting flows.

1.1 First Law of Thermodynamics

The first law of thermodynamics is a version of the law of conservation of energy, adapted for thermodynamic systems. The law of conservation of energy states that the total energy of an isolated system is constant; energy can be transformed from one form to another, but cannot be created or destroyed.

$$\partial W + \partial Q = \partial E + \partial K$$

in specific form

$$\partial w + \partial q = \partial e + \partial \left(\frac{K}{m}\right)$$

1.2 Second Law of Thermodynamics

- Entropy measures the disorder of a system
- It also is a measure of the number of possible states a material can have. s is always maximized, where

  $$s = k \ln(\Omega)$$

  k is the Boltzmann constant. Ω is a measure of the number of possible states
- Entropy for a system must increase

  $$ds = \frac{dq}{T}$$

  dq is an incremental amount of heat added reversibly to a system and T is the system temperature.
- s is a state variable which indicates the direction a process can take
- s can be used for borth reversible and irreversible processes. For reversible processes, $ds = 0$. For irreversible processes, $ds > 0$ (Heat flows from hot to cold).
- Entropy increases in thermodynamic systems.

If we equate dq with the first law,

$$Tds = dq = de + pdv$$

1.2.1 Entropy as a state variable

If we consider a calorically perfect ideal gas,

$$e = c_v T \qquad h = c_p T$$

So then

$$Tds = dq = dh - vdp = c_p dT - RT\frac{dp}{p}$$

and thus

$$\frac{ds}{c_v} = \gamma\frac{dT}{T} - (\gamma - 1)\frac{dp}{p}$$

For calorically perfect gas $\gamma = const$, so

$$\int_{s_1}^{s_2} \frac{1}{c_v} ds = \frac{\Delta s}{c_v} = \gamma \ln\frac{T}{T_0} - (\gamma - 1)\ln\frac{p}{p_0} = \ln\left(\frac{(T/T_0)^\gamma}{(p/p_0)^{\gamma-1}}\right)$$

So

$$\frac{\Delta s}{c_v} = \ln\left(\frac{(T/T_0)^\gamma}{(p/p_0)^{\gamma-1}}\right)$$

This means we can replace T or p with s as a state variable.

1.3 Reversible and Irreversible Processes

- A reversible process is characterized by $\Delta s = 0$
- An irreversible process is characterized by $\Delta s > 0$

Consider internal energy

$$de = dq - pdv$$

Consider an adiabatic process, so that $dq = 0$.

$$de = \left(\frac{\partial e}{\partial v}\right)_T dv + \left(\frac{\partial e}{\partial T}\right)_v dT = -pdv$$

So then recall that $\left(\frac{\partial e}{\partial T}\right)_v = c_v$, so for any gas,

$$\frac{dT}{dv} = -\frac{1}{c_v}\left(\left(\frac{\partial e}{\partial v}\right)_T + p\right)$$

If we consider a thermally perfect gas, $e = e(T)$, so $\left(\frac{\partial e}{\partial v}\right)_T = 0$.

$$\frac{dT}{dv} = -\frac{p}{c_v}$$

Using the ideal gas law,

$$\frac{dT}{dv} = -\frac{RT}{c_v v}$$

Recall that $c_v = \frac{R}{\gamma - 1}$

$$\frac{v}{T}\frac{dT}{dv} = -(\gamma - 1)$$

Similarly,

$$\frac{\rho}{T}\frac{dT}{d\rho} = -\frac{\gamma - 1}{\gamma}$$

$$\frac{v}{p}\frac{dp}{dv} = -\gamma$$

If we further consider a calorically perfect gas, $\gamma = const$

$$\int \frac{dv}{v} = \frac{-1}{\gamma - 1} \int \frac{dT}{T}$$

$$\ln \frac{v}{v_0} = \frac{-1}{\gamma - 1} \ln \frac{T}{T_0}$$

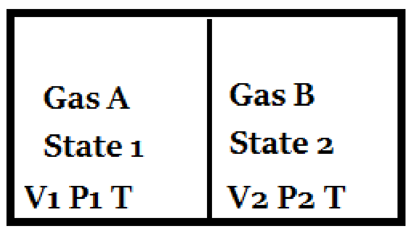
So then for a calorically perfect ideal gas, the isentropic relations are

$$\frac{v}{v_0} = \left(\frac{T}{T_0}\right)^{\frac{-1}{\gamma-1}} \qquad \frac{p}{p_0} = \left(\frac{T}{T_0}\right)^{\frac{\gamma}{\gamma-1}} \qquad \frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

Recall $\frac{\Delta s}{c_v} = \ln\left(\frac{(T/T_0)^{\gamma}}{(p/p_0)^{\gamma-1}}\right)$. If $\Delta s \to 0$, we recover the isentropic relations.

1.3.1 Irreversible Adiabatic Processes

Example : Adiabatic expansion of a gas



We have thermal equilibrium (no Temperature gradients). A pressure gradient created by the removal of the diaphragm will induce fluid motion.

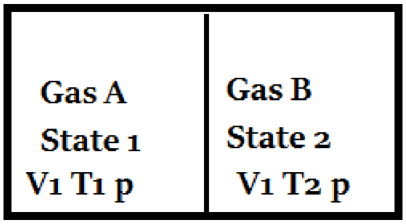
- A shock wave travels into the low pressure region
- Expansion travels into high pressure region
- Viscous and dissipative forces bring fluid into equilibrium

What is the final equilibrium state of the system

$$E_B - E_A = Q + W$$

For this system , $Q = W = 0$, so $E_B = E_A$.
If we assume the gas is perfect, then $E = E(T)$ and so $E_B(T_B) = E_A(T_A)$ and thus $T_B = T_A = T$. What is the final pressure? (see HW)

Example: Insulated vessel



If the partition is removed, there is a temperature gradient. This produces an energy gradient (aka current). $\nabla T = 0$ when we reach steady state. First law says

$$E_B - E_A = Q + W$$

We define the system as the entire box. Thus as before $Q = W = 0$ and $E_B = E_A$.

$$E_A = M_1 e(T_1, p) + M_2 e(T_2, p)$$

but also

$$E_B = (M_1 + M_2) e(T_B, p)$$

so then

$$M_1 e(T_1, p) + M_2 e(T_2, p) = (M_1 + M_2) e(T_B, p)$$

and thus

$$e(T_B, p) = \frac{M_1}{M_1 + M_2} e(T_1, p) + \frac{M_2}{M_1 + M_2} e(T_2, p)$$

If we assume a calorically perfect gas, $e = c_v T$ and thus

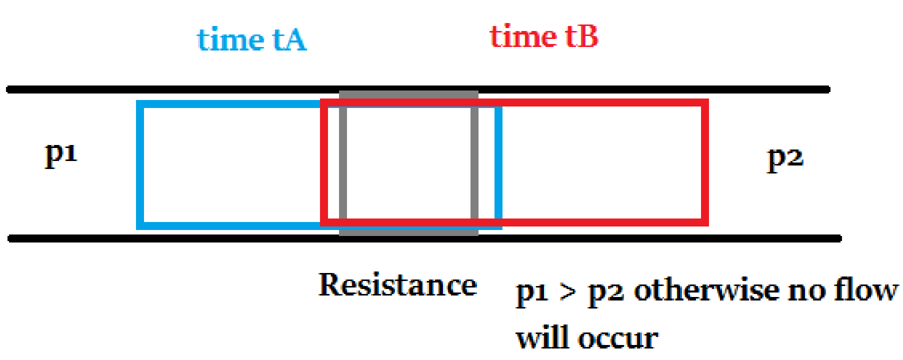
$$e(T_B, p) = c_{v_B} T_B = \frac{M_1}{M_1 + M_2} c_{v_1} T_1 + \frac{M_2}{M_1 + M_2} c_{v_2} T_2$$

If the gases were sufficiently similar (molecular weight, c_V, R, etc) in each part of the gas, then

$$T_B = \frac{M_1}{M_1 + M_2} T_1 + \frac{M_2}{M_1 + M_2} T_2$$

See HW1 for when the gases are different.

Example: Flow Throttling Consider an insulated pipe/channel with resistance inside. Assume the kinetic energy is negligible (the usual incompressible flow assumption) compared to the internal energy $(\frac{1}{2} u^2 << e)$



We track the control volume of a fluid from state A to state B. We write the 1st law noting that no heat is exchanged.

$$\Delta E = Q + W = W$$

The work to push the fluid through the resistance is the flow work. Let blue be volume 1 and let red be volume 2.

$$W_{flow} = \int_1 pdV - \int_2 pdV = \int_1 p_1 dV - \int_2 p_2 dV$$

We assume that the pressure does not vary spatially

$$W_{flow} = p_1 \int_1 dV - p_2 \int_2 pdV = p_1 V_1 - p_2 V_2$$

So then the first law is

$$E_2 - E_1 = p_1 V_1 - p_2 V_2$$

or

$$e_2 - e_1 = p_1 v_1 - p_2 v_2$$

Noting that $h = e + pv$,

$$h_1 = h_2$$

If heat is added,

$$h_1 = h_2 + q$$

2 Steady One Dimensional Flow

2.1 Continuity Equation (Conservation of Mass)

Consider a steam tube The flow is everywhere tangent to the tube: $\vec{v} \cdot \vec{n} = 0$. Consider a small fluid volume over a small distance Δx.

$$\text{Inflow} = \rho u A$$

$$\text{Outflow} = \rho u A + \frac{d}{dx}(\rho u A)\Delta x$$

The rate of change of the mass inside of the volume will be

$$\frac{d}{dt}(\rho A \Delta x)$$

Net mass flow through the tube is

$$\frac{d}{dt}(\rho A \Delta x) = \rho u A - \left(\rho u A + \frac{d}{dx}(\rho u A)\Delta x\right)$$

We simplify this to the continuity equation

$$\frac{d}{dt}(\rho A) + \frac{d}{dx}(\rho u A) = 0$$

For steady flow, there is no time variation. So the above becomes

$$\frac{d}{dx}(\rho u A) = 0$$

or rather

$$\rho u A = const$$

2.2 Conservation of Energy

If heat is added along a stream tube,

$$h_2 = h_1 + q + w$$

For compressible flows, we need to consider the kinetic energy. Thus the total enthalpy is

$$h_{total} = h_T = h + \frac{1}{2}u^2$$

So the energy increase from heat added,

$$h_1 + \frac{1}{2}u_1^2 + q = h_2 + \frac{1}{2}u_2^2$$

or we could write this as

$$q = h_{T_2} - h_{T_1}$$

This is true as long as states 1 and 2 are in equilibrium. That is, this is valid even if you have viscous stresses or heat transfer or other non-equilibrium conditions in between state 1 and 2.
For $q = 0$,

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

and thus for an adiabatic process,

$$h_{T_2} = h_{T_1}$$

If we assume that the flow is in equilibrium everywhere then

$$h_T = h + \frac{1}{2}u^2 = const$$

along the stream tube. Differentiating,

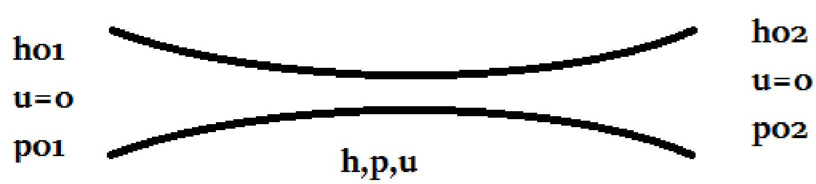
$$\partial h + u \partial u = 0$$

If a gas is thermally perfect where $h = h(T)$,

$$c_p \partial T + u \partial u = 0$$

2.3 Stagnation Conditions

Recall $h_T = h + \frac{1}{2}u^2$. When a flow is slowed down to zero velocity, we recover the stagnation enthalpy $h_0 = h$. Consider a tube



Fluid will flow if $p_{0_1} > p_{0_2}$.

- For natural processes, $ds \geq 0$ for flow to be induced
- $h_{0_2} - h_{0_1} = q =$ heat added

2.4 Euler's Equations

- Lagrangian formulations consider the particle reference frame
- Eulerian formulations consider the lab/fixed reference frame

Recall the material derivative in 1D $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$ (valid in the Lagrangian reference frame). The acceleration can be expressed as

$$a = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}$$

In this case, time derivatives are known as the non-stationary terms and the space derivatives are known as the convective terms. As before, we recall the net force on a particle is

$$F = pA - \left(pA + \frac{\partial}{\partial x}\left(pA\Delta x\right)\right) = -\frac{\partial}{\partial x}\left(pA\Delta x\right)$$

The Body Force acting on any arbitrarily shaped particle

$$f = -\frac{\partial p}{\partial x} \qquad f = \frac{F}{\Delta x A}$$

We could also integrate the pressure over the surface

$$\bar{F} = \int_S p d\bar{s} = \int_V \nabla p dv$$

For quasi-1D flow we see

$$\int_S p\hat{x}\cdot d\bar{s} = \int_V \hat{x}\cdot\nabla p dv = \int_V \frac{\partial p}{\partial x} dv$$

So then we can write

$$F = ma \implies f = \rho a \implies -\frac{\partial p}{\partial x} = \rho\frac{Du}{Dt}$$

Thus we have

$$-\frac{\partial p}{\partial x} = \rho\frac{\partial u}{\partial t} + u\rho\frac{\partial u}{\partial x}$$

or

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

For steady flow $\frac{\partial u}{\partial t} = 0$, and we can write

$$\partial p = -\rho u \partial u$$

In integral form we have Bernoulli's equation for compressible flow

$$\frac{u^2}{2} + \int\frac{\partial p}{\rho} = const$$

For incompressible flows, $\rho = \rho_0 = const$. We have Bernoulli's eqn

$$\frac{1}{2}\rho_0 u^2 + p = const$$

2.5 Conservation of Momentum

- Primitive variables are density, pressure, and velocity.
- Conserved Variables are density, total Energy, and momentum

$$\rho A \frac{\partial u}{\partial t} + \rho u A \frac{\partial u}{\partial x} = -A \frac{\partial p}{\partial x}$$

Multiply the continuity by u

$$u \frac{\partial u}{\partial t}(\rho A) + u \frac{\partial}{\partial x}(\rho u A) = 0$$

Add these together to get the Momentum Equation for unsteady quasi-1D flow

$$\frac{\partial}{\partial t}(u \rho A) + \frac{\partial}{\partial x}\left(\rho u^2 A\right) = -\frac{\partial}{\partial x}(pA) + p\frac{\partial A}{\partial x}$$

In integral form, (integrate in space)

$$\frac{\partial}{\partial t}\int_A^b (u \rho A)\, dx + \left[\rho u^2 A\right]_a^b = \left[-pA\right]_a^b + \int_a^b p\frac{\partial A}{\partial x} dx$$

The integral form is valid even when friction, dissipative forces, or irreversible processes are present as long as they are within the control volume. The fluid must be in equilibrium at the control surfaces A and B.

In 1D flow, A is constant. Thus we have the conservative form

$$\frac{\partial}{\partial t}(u\rho) + \frac{\partial}{\partial x}\left(\rho u^2 + p\right) = 0$$

For steady 1D flow,

$$\rho u^2 + p = const$$

or as Euler's eqn

$$dp = -\rho u du$$

This assumes the fluid is in equilibirum everywhere. Or we can state

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

In this case only states 1 and 2 need to be in equilibrium.

2.6 Summary of Governing Equations

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2$$

$$p_1 A_1 + \rho_1 u_1^2 A_1 + \int_1^2 p dA = p_2 A_2 + \rho_2 u_2^2 A_2$$

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

For 1D,

$$\rho_1 u_1 = \rho_2 u_2$$

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$$

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

3 Isentropic Flow

Consider a adiabatic reversible flow. $ds = 0$ Thus the differential form of equations are valid everywhere.

$$d(\rho u) = 0$$

$$\rho u du + dp = 0$$

$$dh + udu = 0$$

We then consider the reference frame of a wave moving at speed a, then see that conservation of mass across the wave gives

$$\rho a = (\rho + d\rho)(a + da) \implies \rho da + a d\rho + \text{second order terms} = 0 \implies a = -\rho \frac{\partial a}{\partial \rho}$$

Conservation of momentum gives us

$$\rho a^2 + p = (\rho + \partial\rho)(a + \partial a)^2 + (p + \partial p) \implies 0 = dp + 2\rho a \partial a + a^2 \partial\rho \implies \frac{\partial a}{\partial \rho} = \frac{\frac{\partial p}{\partial \rho} + a^2}{-2a\rho}$$

So then

$$a = -\rho \frac{\frac{\partial p}{\partial \rho} + a^2}{-2a\rho} \implies a^2 = \frac{\partial p}{\partial \rho}$$

Recall we assumed reversible flow with small perturbations, so

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

For isentropic flow for calorically perfect gas $pv^\gamma = const$ or $p = c\rho^\gamma$ for some c, so

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \gamma c \rho^{\gamma-1} = \frac{\gamma p}{\rho}$$

Thus for a calorically perfect gas,

$$a = \sqrt{\frac{\gamma p}{\rho}}$$

If we assume an ideal gas and use the ideal gas law,

$$a^2 = \gamma RT$$

3.1 Mach Number

The Mach number is a dimensionless speed of the sound

$$M = \frac{u}{a}$$

- Subsonic $M < 1$
- Sonic $M = 1$
- Supersonic $M > 1$
- Hypersonic $M < 5$

We have two measures of the relative importance of compressibility of a flow

$$\frac{\text{kinetic energy}}{\text{internal energy}} = \frac{\frac{1}{2}u^2}{c_v T} = \frac{\frac{1}{2}u^2}{\frac{RT}{\gamma-1}} = \frac{\gamma(\gamma-1)u^2}{2a^2} = \frac{\gamma(\gamma-1)M^2}{2}$$

OR

$$\frac{\text{kinetic energy}}{\text{enthalpy}} = \frac{u^2}{2h} = \frac{\gamma-1}{2}M^2$$

Notice that M^2 is involved in both ratios.

Consider the area-volume relations

$$\partial\left(\rho u A\right) = 0 \implies \frac{\partial A}{A} + \frac{\partial u}{u} + \frac{\partial \rho}{\rho} = 0$$

Recall Euler's eqn

$$dp = -\rho u \partial u = \frac{\partial p}{\partial \rho}\partial\rho = -\rho u \partial u \implies \frac{\partial p}{\partial \rho}\frac{\partial \rho}{\rho} = -u\partial u$$

If we assume isentropic flow,

$$\frac{\partial p}{\partial \rho} = a^2 \implies a^2\frac{\partial \rho}{\rho} = -u\partial u$$

We rewrite to get

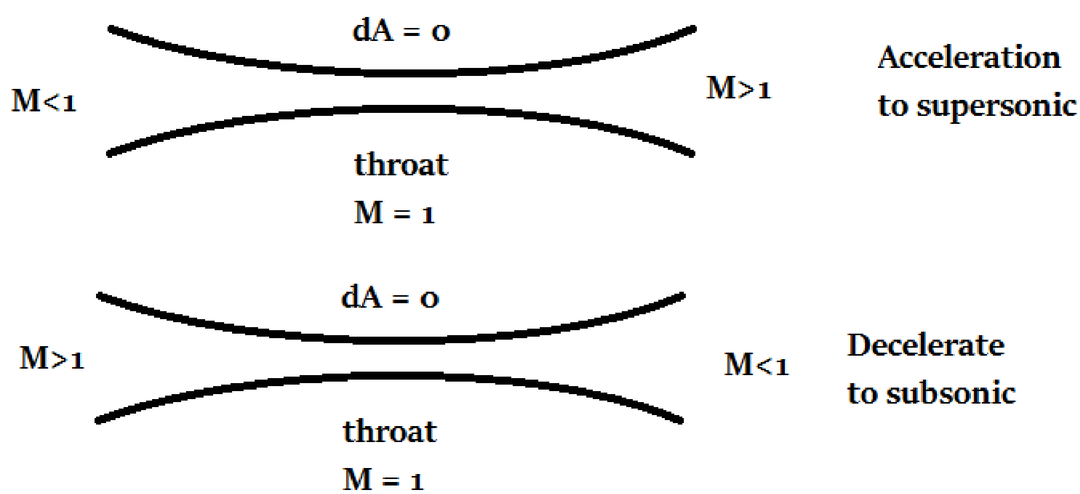
$$\frac{\partial \rho}{\rho} = \frac{-u\partial u}{a^2} = -M^2\frac{\partial u}{u}$$

so then

$$\frac{\partial A}{A} = (M^2-1)\frac{\partial u}{u}$$

This tells us about the velocity change given an area change.

- As $M \to 0$, $uA = const$ and we have the incompressible limit
- For $0 < M < 1$, $\partial A > 0$ yields $\partial u < 0$. Increase in area leads to a decrease in velocity and vice versa.
- $M = 1$, $\partial A = 0$. This implies that we have a maximum or a minimum in the area, ie a convergent-divergent nozzle.



- $M > 1$, $\partial A > 0$ yields $\partial u > 0$. Increase in area leads to an increase in velocity and vice versa.

3.2 Mach Flow

Sound speed acts a measure of thermal energy. We can define a location in a flow where $M = 1$ as the sonic point

$$\frac{a^2}{\gamma-1} + \frac{u^2}{2} = \frac{a^{*2}}{\gamma-1} + \frac{a^{*2}}{2} = \frac{\gamma+1}{2(\gamma-1)}a^{*2}$$

If the flow field is adiabatic (need not be reversible) then a^* is constant everywhere.

Recall total/stagnation condition from energy conservation

$$c_pT + \frac{u^2}{2} = c_pT_0 \implies \frac{T_0}{T} = 1 + \frac{u^2}{2\gamma RT/(\gamma-1)} = 1 + \frac{u^2}{2a^2/(\gamma-1)} = 1 + \frac{\gamma-1}{2}M^2$$

3.2.1 Stagnation Relationships

If we desire stagnation relationships for pressure and density, we must assume the flow is adiabatic AND reversible (isentropic).

$$\frac{\rho_0}{\rho} = \left(\frac{T_0}{T}\right)^{\frac{1}{\gamma-1}} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{1}{\gamma-1}}$$

$$\frac{p_0}{p} = \left(\frac{\rho_0}{\rho}\right)^{\gamma} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}$$

These give you the stagnation conditions in isentropic flow. That is, at a stagnation point (zero fluid flow), the T, ρ, p of a fluid is T_0, ρ_0, p_0

We see now that we are able to relate the stagnation and sonic point conditions

$$\frac{\gamma+1}{2(\gamma-1)}a^{*2} = \frac{a_0^2}{\gamma-1}$$

At $M = 1$

$$\frac{a^*}{a_0} = \frac{T^*}{T_0} = \frac{2}{\gamma+1} \text{ (ratio is constant)}$$

$$\frac{p^*}{p_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho^*}{\rho_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}$$

Sometimes it is convenient to define the speed

$$\frac{u}{a^*} = M^* \text{ (this is not the Mach number)}$$

So then

$$\frac{a^2}{\gamma-1} + \frac{u^2}{2} = \frac{\gamma+1}{2(\gamma-1)}(a^*)^2$$

Divide by u

$$\frac{1}{\gamma-1}\left(\frac{a}{u}\right)^2 + \frac{1}{2} = \frac{\gamma+1}{2(\gamma-1)}\left(\frac{a^*}{u}\right)^2$$

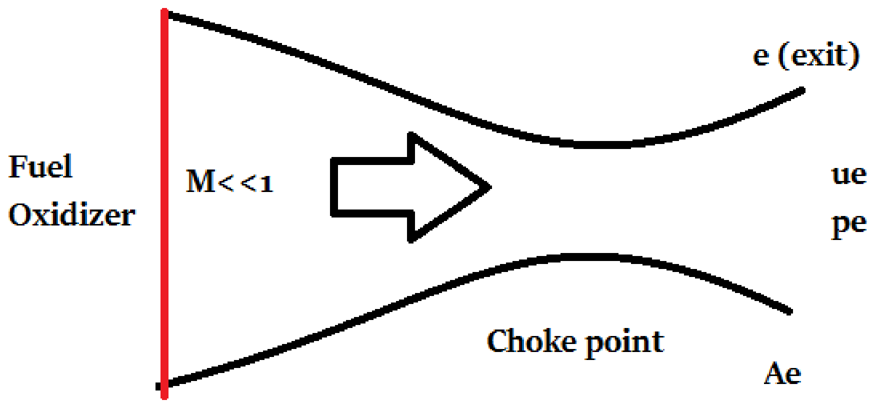
So then

$$M^* = \frac{\gamma+1}{\frac{2}{M^2} + \gamma - 1}$$

- Note $M > 1 \iff M^* > 1$
- Note $M < 1 \iff M^* < 1$
- As $M \to \infty$, $M^* \to \frac{\gamma+1}{\gamma-1} < \infty$

3.3 Rocket Nozzle

Assume we have an isentropic flow of a calorically perfect gas.



$Pc = 15atm, Tc = 2500K, cp = 4157J/kg/K, Te = 1350K, W = 12kg/kmol$ (molecular weight). We need to find the exit pressure, velocity, M.
Since $M << 1$ in the chamber $p_0 = p_c$ and $T_0 = T_c$. Now we need γ

$$R = \frac{R_u}{W} = \frac{8314J/kmol/K}{12kg/kmol} = 692.8J/kg/K$$

Also

$$c_v = c_p - R = 3464J/kg/K$$

Thus

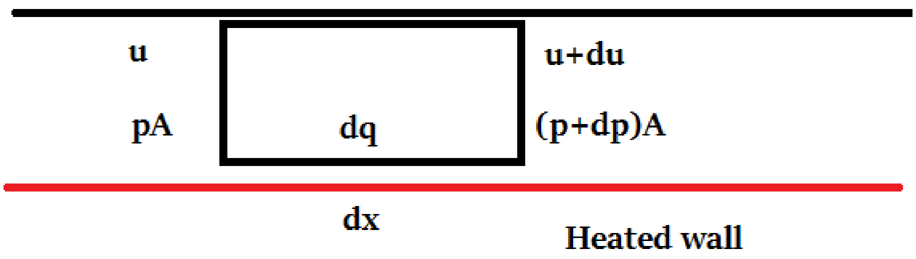
$$\gamma = \frac{c_p}{c_v} = 1.2$$

We assume γ is constant under temperature interval and use isentropic flow relations.

3.4 Rayleigh Flow

- 1D flow with heat
- Frictionless flow , ie close to reversible

How does flow change when heat is added? Sources could be combustion, conduction, heat transfer, evaporation, or condensation.



$$d(\rho u) = 0$$

$$dp + \rho u du = 0$$

$$dh + u du = 0$$

Use the ideal gas law $p = \rho RT$

$$dp = RT d\rho + \rho R dT$$

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

$$c_p dT + u du = dq$$

If we have a calorically perfect gas, this becomes

$$d(c_p dT + \frac{1}{2}u^2) = c_p dT_0 = dq$$

Thus change in energy implies change in total energy

$$ds = c_p \frac{dT}{T} - R\frac{dp}{p}$$

where $ds \geq \frac{dq}{T}$. If ∇T is small, we can regard the flow as nearly reversible $ds = \frac{dq}{T}$.

$$d(M^2) = d\left(\frac{u^2}{\gamma RT}\right)$$

$$dM = \frac{du}{a} - \frac{M}{2}\frac{dT}{T}$$

$$\frac{dM}{M} = \frac{du}{u} - \frac{dT}{2T}$$

Use local stagnation T, p ratios

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2}M^2$$

$$\frac{p_0}{p} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{\frac{\gamma}{\gamma - 1}}$$

Combine all of these and use $dT_0 = \frac{dq}{c_p}$.

$$\frac{dM}{M} = \frac{\left(1+\gamma M^2\right)\left(1+\frac{\gamma-1}{2}M^2\right)}{2\left(1-M^2\right)}\frac{dT_0}{T_0}$$

$$\frac{du}{u} = -\frac{d\rho}{\rho} = \frac{\left(1+\frac{\gamma-1}{2}M^2\right)}{\left(1-M^2\right)}\frac{dT_0}{T_0}$$

Remember $dq > 0 \iff dT_0 > 0$ and $dq < 0 \iff dT_0 < 0$

$$\frac{\partial p_0}{p_0} = \frac{-\gamma^2 M^2}{2}\frac{\partial T_0}{T_0}$$

$$\frac{\partial s}{c_p} = \left(1+\frac{\gamma-1}{2}M^2\right)\frac{\partial T_0}{T_0}$$

3.4.1 Impact on Flows

For subsonic flow, $M < 1$, so if we have head addition $\partial q > 0$,

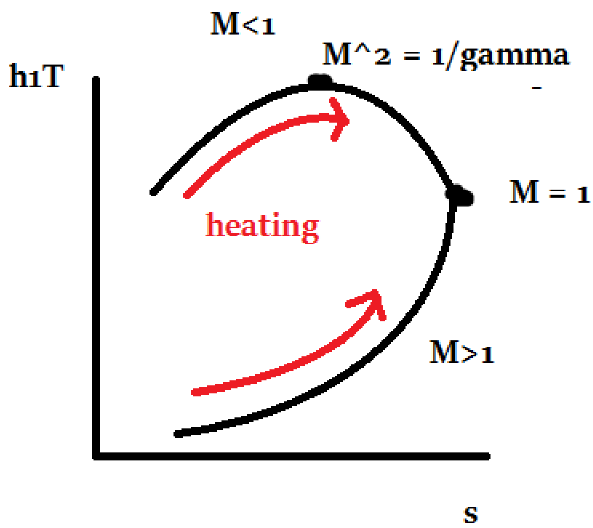
- $\partial T_0 > 0$
- $\partial M > 0$
- $\partial p < 0$
- $\partial u > 0$
- $\partial \rho < 0$
- $\partial p_0 < 0$
- $\partial s > 0$
- $\partial T > 0$ for $M^2 < \frac{1}{\gamma}$, $\partial T < 0$ for $M^2 > \frac{1}{\gamma}$

For heat removal $\partial q < 0$, the inequalities are flipped.

For supersonic flow, $M > 1$. If we have head addition $\partial q > 0$,

- $\partial T_0 > 0$
- $\partial M < 0$
- $\partial p > 0$
- $\partial u < 0$
- $\partial \rho > 0$
- $\partial p_0 < 0$
- $\partial s > 0$
- $\partial T > 0$

For heat removal $\partial q < 0$, the inequalities are flipped.



- If you are in a subsonic flow, heating will start cooling the flow past the maximum because then the heat will become kinetic energy instead of thermal energy.
- You cannot heat a gas past $M = 1$
- Heat drives the flow towards sonic conditions $M = 1$
- Cooling drives the flow away from sonic conditions
- Mathematically you could transition from $M < 1$ to $M > 1$ through heat transfer
- Heating always decreases total pressure
- For $\frac{1}{\gamma} < M^2 < 1$, heating decreases the temperature since heat converts to kinetic instead of thermal energy

We can integrate to obtain relations relative to the sound speed reference state $M = 1$

$$\frac{p}{p^*} = \frac{\gamma + 1}{1 + \gamma M^2}$$

$$\frac{\rho}{\rho^*} = \frac{v^*}{v} = \frac{u^*}{u} = \frac{1 + \gamma M^2}{(1 + \gamma) M^2}$$

$$\frac{T}{T^*} = \frac{(\gamma + 1)^2 M^2}{(1 + \gamma M^2)^2}$$

$$\frac{p_0}{p_0^*} = \frac{\gamma + 1}{1 + \gamma M^2} \left(\frac{2}{\gamma + 1} \left(1 + \frac{\gamma - 1}{2} M^2 \right) \right)^{\frac{\gamma}{\gamma - 1}}$$

$$\frac{T_0}{T_0^*} = \frac{2(\gamma + 1) M^2}{(1 + \gamma M^2)^2} \left(1 + \frac{\gamma - 1}{2} M^2 \right)$$

NOTE, $p_{total} = p_0$ only at an actual stagnation point. Otherwise p_0 is merely a convenient quantity. This is also true for other stagnation variables.

To find a new state 2 from state 1, with reference state $*$ at $M = 1$,

4 Friction

Over short distances, flow can be isentropic (reversible) flow. Over long distances, we must consider friction which gives rise to non-isentropic (irreversible flow).
If a duct has perimeter P and a non-varying cross sectional area A, let τ_w be the wall shear stress. Then, conservation of mass tells us

$$\partial(\rho u A) = 0 \implies \partial(\rho u) = 0$$

Conservation of momentum tells us

$$\partial(\rho u^2 + pA) = p\frac{\partial A}{\partial x} + forces \implies uA\partial(\rho u) + \rho u A \partial u + A\partial p = -\tau_w P \partial x$$

By conservation of mass, $\partial(\rho u) = 0$. Now let $D_H = \frac{4A}{P}$ be the hydraulic diameter. So we have

$$\rho u \partial u + \partial p + \frac{4\tau_w}{D_H}\partial x = 0$$

Thus the fluid momentum is not conserved. Now we consider the conservation of energy,

$$\partial h + u\partial u = 0$$

Since the flow is irreversible due to shear stresses, we know $\Delta s > 0$.

4.1 Ideal Gas with friction

If we assume an ideal gas, we can dig deeper.

$$\frac{\partial p}{p} = \frac{\partial \rho}{\rho} + \frac{\partial T}{T}$$

$\partial h = c_p \partial T$, so $\partial s = c_p \frac{\partial T}{T} - R\frac{\partial p}{p}$ and so

$$\frac{\partial M}{M} = \frac{\gamma M^2 \left(1 + \frac{\gamma - 1}{2} M^2\right)}{1 - M^2}\left(\frac{4\tau_w}{\rho u^2 D_H}\partial x\right)$$

$$\frac{\partial u}{u} = \frac{\gamma M^2}{2(1 - M^2)}\left(\frac{4\tau_w}{\rho u^2 D_H}\partial x\right)$$

$$\frac{\partial T}{T} = \frac{-\gamma(\gamma - 1)M^4}{1 - M^2}\left(\frac{4\tau_w}{\rho u^2 D_H}\partial x\right)$$

$$\frac{\partial p}{p} = \frac{-\gamma M^2 \left(1 + (\gamma - 1)M^2\right)}{1 - M^2}\left(\frac{4\tau_w}{\rho u^2 D_H}\partial x\right)$$

$$\frac{\partial \rho}{\rho} = \frac{-\gamma M^2}{1 - M^2}\left(\frac{4\tau_w}{\rho u^2 D_H}\partial x\right)$$

$$\frac{\partial s}{c_p} = \frac{(\gamma - 1)M^2}{2}\left(\frac{4\tau_w}{\rho u^2 D_H}\partial x\right)$$

Recall stagnation pressure as pressure if flow was brought to rest.

$$\frac{p_0}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}}$$

We can differentiate and rearrange this

$$\frac{\partial p_0}{p} = \frac{-\gamma M^2}{2} \left(\frac{4\tau_w}{\rho u^2 D_H} \partial x\right)$$

You could also show this by

$$\partial s = c_p \frac{\partial T_0}{T_0} - R \frac{\partial p_0}{p_0}$$

In an adiabatic flow $\partial T_0 = 0$ since at rest there is no friction. So

$$\frac{\partial p_0}{p_0} = -\frac{\partial s}{R} = -\frac{c_p}{R} \frac{1 - \gamma}{2} M^2 \left(\frac{4\tau_w}{\rho u^2 D_H} \partial x\right) = \frac{-\gamma M^2}{2} \left(\frac{4\tau_w}{\rho u^2 D_H} \partial x\right)$$

4.2 Qualitative Results

$\tau_w > 0$ since it is a drag force.
In a subsonic flow,

- $\partial M > 0$
- $\partial u > 0$
- $\partial T < 0$
- $\partial p < 0$
- $\partial \rho < 0$
- $\partial s > 0$
- $\partial T_0 = 0$
- $\partial p_0 < 0$

In a supersonic flow, some of these terms are flipped.

We normally write shear stress in terms of a friction factor f

- Fanning friction $\tau_w = f_F \frac{1}{2} \rho u^2$ (mostly used for compressible flows)
- D'Arcy Friction $\tau_w = f_D \frac{1}{8} \rho u^2$ (mostly used for incompressible flows)
- In general $f = f(\mathcal{R}_e, \frac{\epsilon}{D_H}, M)$. Note that the Mach number effects are usually small in comparison to the Reynolds number and surface roughness effects. If $\mathcal{R}_e$ is very large, friction might be insensitive to $\frac{\epsilon}{D_H}$.
- Friction drives flow towards $M = 1$ irreversibly. That means the arrows on the Fanno Curve are only pointing towards $M = 1$ and will not point away. We cannot undo the effects of friction.

4.3 Fanno Flow

Using fanning friction,

$$\frac{4\tau_w}{\rho u^2 D_H}\partial x = \frac{2f}{D_H}\partial x$$

So then

$$\frac{\partial M}{M} = \frac{\gamma M^2\left(1+\frac{\gamma-1}{2}M^2\right)}{1-M^2}\frac{2f}{D_H}\partial x$$

So then

$$\int_{M_1}^{M_2} 2\frac{1-M^2}{\gamma M^2\left(1+\frac{\gamma-1}{2}M^2\right)}\frac{\partial M}{M} = \frac{4f}{D_H}\int_{x_1}^{x_2}\partial x$$

This gives us

$$\frac{1}{\gamma}\left(\frac{1}{M_1^2}-\frac{1}{M_2^2}\right)+\frac{\gamma+1}{2\gamma}\log\left(\frac{M_1^2}{M_2^2}\right)^2\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2} = \int_{x_1}^{x_2}\frac{4f}{D_H}\partial x$$

Recall that if $T_{01} = T_{02}$

$$\frac{T_2}{T_1} = \frac{T_0}{T_1}\frac{T_2}{T_0} = \frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}$$

$$\frac{p_2}{p_1} = \frac{M_1}{M_2}\left(\frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}\right)^{\frac{1}{2}}$$

$$\frac{\rho_2}{\rho_1} = \frac{M_1}{M_2}\left(\frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}\right)^{-\frac{1}{2}}$$

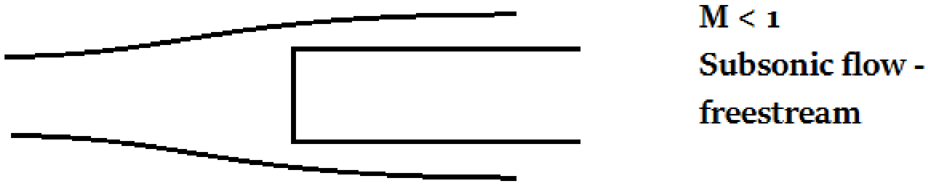
$$\frac{p_{02}}{p_{01}} = \frac{M_1}{M_2}\left(\frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}$$

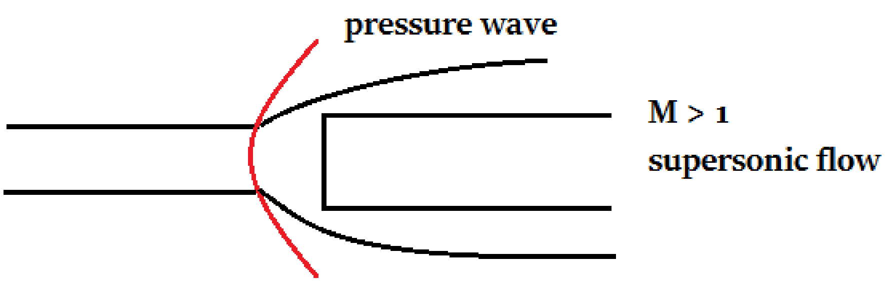
We can write these with respect to a common reference condition $M = 1$. Define a reference length $x = L^*$ for $M = 1$, so then

$$\int_{x_1}^{x_2}\frac{4f}{D_H}\partial x = \frac{4f}{D_H}L^*\bar{f} = \frac{1-M^2}{\gamma M^2}+\frac{\gamma+1}{2\gamma}\log\left(\frac{(\gamma+1)M^2}{2+(\gamma-1)M^2}\right)$$

where M is the mach number at $x = 0$ and $\bar{f}$ is the average friction coefficient.

5 Normal Shocks





Finite disturbances coalesce to form a shock wave. If we consider the adiabatic flow across a sock wave, mass momentum and energy are conserved.
In 1D, we can consider the states before and after shocks

$$\rho_1 u_1 = \rho_2 u_2$$

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$$

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2$$

5.1 Missing notes

5.2 Stagnation Pressure Decrease Across a Shock Wave

Write $p_{0_2} = p_{0_1} + \Delta p_{0_1}$

$$\frac{s_2 - s_1}{R} = -\log\frac{p_{0_1} + \Delta p_{0_1}}{p_{0_1}} = -\log\left(1 + \frac{\Delta p_{0_1}}{p_{0_1}}\right) = -\left(\frac{\Delta p_{0_1}}{p_{0_1}} - \left(\frac{\Delta p_{0_1}}{p_{0_1}}\right)^2 + ...\right) \approx -\frac{\Delta p_{0_1}}{p_{0_1}}$$

for a weak shock. That is, the entropy increase is directly proportional to the pressure change. We also note that

$$\frac{s_2 - s_1}{R} = \frac{2\gamma}{(\gamma+1)^2}\frac{(M_1^2 - 1)^3}{6} \approx -\frac{\Delta p_{0_1}}{p_{0_1}}$$

This is a very different relationship than for the static pressure difference. This is beneficial for engines since we can have compression without a loss of stagnation pressure by using a series of weak shocks.

$$\frac{p_0}{p} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{p_{0_2}}{p_{0_1}} = \frac{p_2}{p_1}\left(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right)^{\frac{\gamma}{\gamma-1}}$$

We know $\frac{p_2}{p_1} = f(M_1)$, $M_2 = M_2(M_1)$, so we use this to obtain

$$\frac{p_{0_2}}{p_{0_1}} = \left(1+\frac{2\gamma}{\gamma+1}\left(M_1^2-1\right)\right)^{-\frac{1}{\gamma-1}}\left(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right)^{-\frac{\gamma}{\gamma-1}}$$

or

$$\frac{p_{0_2}}{p_{0_1}} = \left(1+\frac{2\gamma}{\gamma+1}\left(M_1^2-1\right)\right)^{-\frac{1}{\gamma-1}}\left(\frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2}\right)^{\frac{\gamma}{\gamma-1}}$$

5.3 Rakine-Hugoniot Relations

We can express jump conditions in terms of a single jump condition , so we can write jump conditions without knowledge of M_1 or u_1

$$\frac{\rho_2}{\rho_1} = f\left(\frac{p_2}{p_1}\right)$$

For continuity,

$$u_2 = u_1\left(\frac{\rho_1}{\rho_2}\right)$$

Sub this into the momentum eqn

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2\left(\frac{\rho_1 u_1}{\rho_2}\right)^2$$

or

$$\frac{p_2-p_1}{v_2-v_1} = -\left(\frac{u_1}{v_1}\right)^2$$

Similarly

$$u_2^2 = \frac{p_2-P_1}{\rho_2-\rho_1}\left(\frac{\rho_1}{\rho_2}\right)$$

Insert this into the energy equation

$$e_1 + \frac{p_1}{\rho_1} + \frac{u_1^2}{2} = e_2 + \frac{p_2}{\rho_2} + \frac{u_2^2}{2}$$

so then this simplifies to

$$e_2 - e_1 = \frac{p_1+p_2}{2}\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)$$

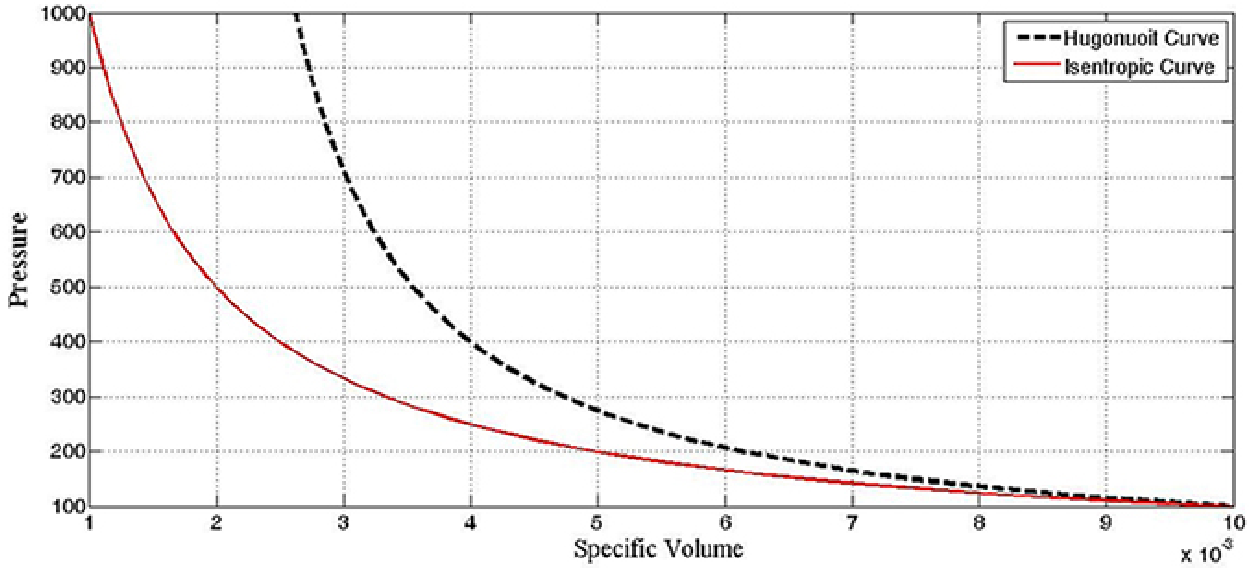
or

$$e_2 - e_1 = \frac{p_1+p_2}{2}\left(v_1-v_2\right)$$

or

$$h_2 - h_1 = \frac{v_1+v_2}{2}\left(p_1-p_2\right)$$

Equations are true in general, valid for any $e = e(p, v)$. This allows us to find state 2 from original state 1. This gives rise to Hugoniot curves



- Without further constraints one cannot simultaneously raise p and v

$$\frac{p_2 - p_1}{v_2 - v_1} = -\left(\frac{u_1}{v_1}\right)^2 < 0$$

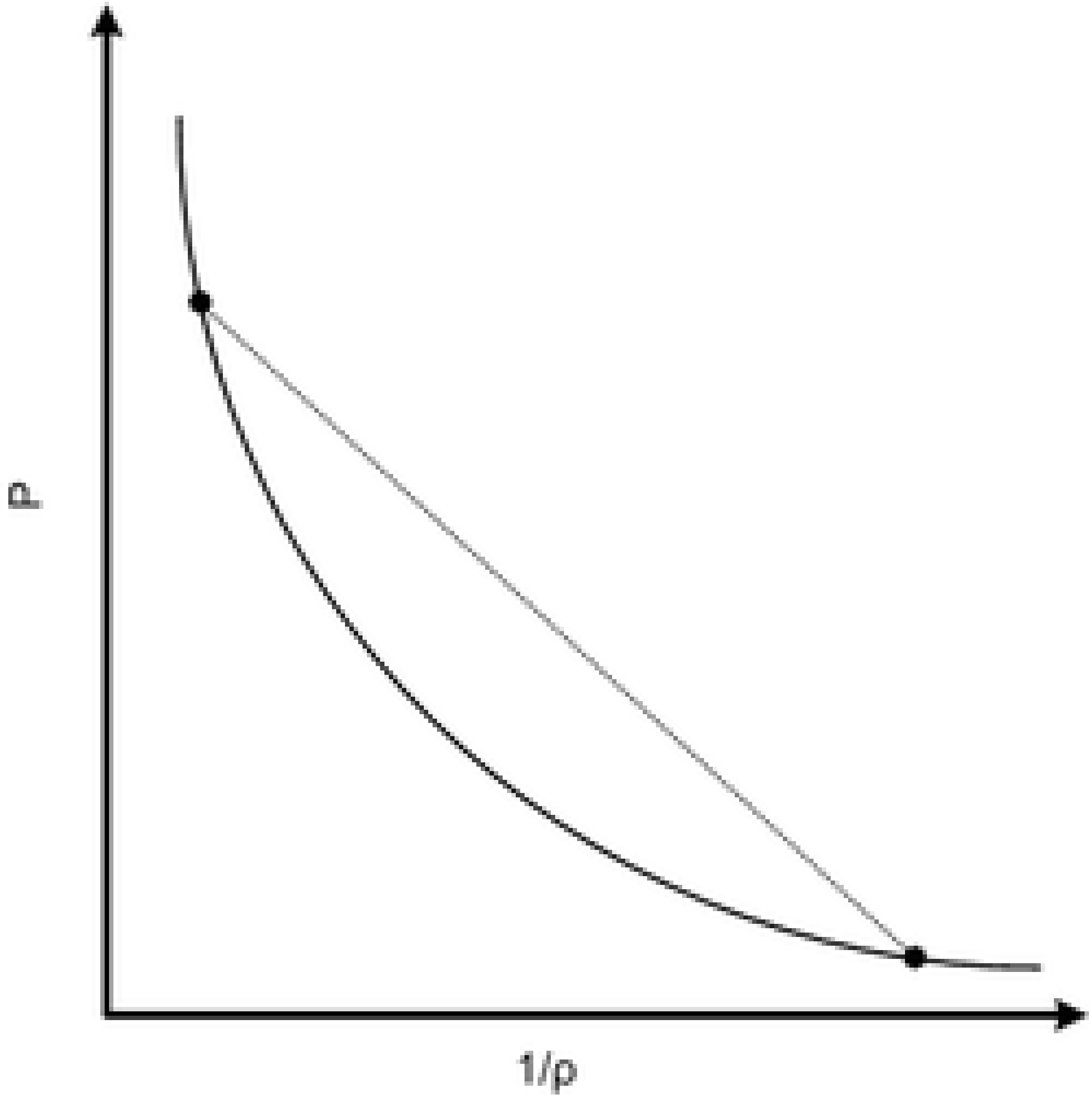
  This relationship has a negative slope

- $p_2 > p_1$ from the second law. That is- we would have to violate entropy ii order to create an expansion shock $p_2 < p_1$. These expansion shocks do not occur in nature but they do occur in CFD codes without entropy fixes.

- The Hugonoit curve represents the locus of all possible states behind normal shocks for some (p, v). Each point corresponds to different upstream velocities u_1. Find a line with slope m at point p_1, v_1 where

$$m = \frac{p_2 - p_1}{v_2 - v_1} = -\left(\frac{u_1}{v_1}\right)^2$$

The intersection of this line and the Hugonoit curve (besides (p_1, v_1) indicates the new state.



This works for any equation of state

- If we assume a perfect gas $e = c_v T$ and $p = \rho R T$ we can write down a closed form solution.

$$\frac{p_2}{p_1} = \frac{\frac{\gamma+1}{\gamma-1}\frac{v_1}{v_2} - 1}{\frac{\gamma+1}{\gamma-1} - \frac{v_1}{v_2}}$$

We can also find $\frac{T_2}{T_1} = f\left(\frac{v_1}{v_2}\right)$ or $\frac{v_2}{v_1} = f\left(\frac{p_2}{p_1}\right)$.

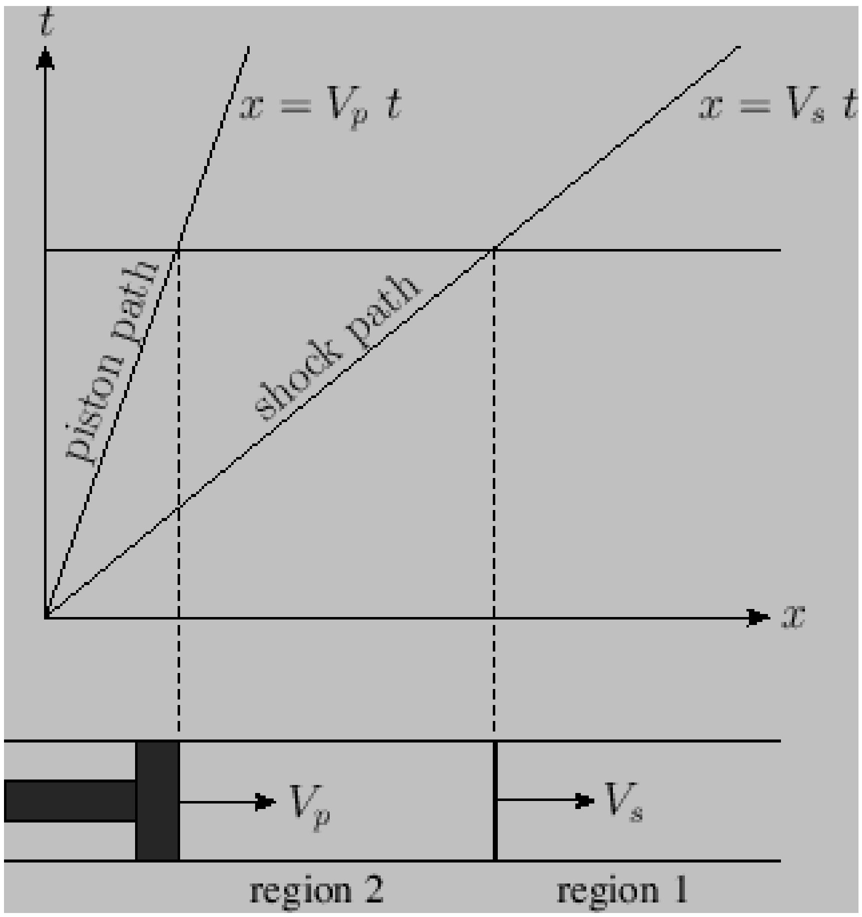
$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\frac{\gamma+1}{\gamma-1} + \frac{p_2}{p_1}}{\frac{\gamma+1}{\gamma-1}\frac{p_2}{p_1} + 1}$$

$$\frac{u_1}{u_2} = \frac{\rho_2}{\rho_1} = \frac{\frac{\gamma+1}{\gamma-1}\frac{p_2}{p_1} + 1}{\frac{\gamma+1}{\gamma-1} + \frac{p_2}{p_1}}$$

Part II
Wave Motion

6 1D Wave motion

Previously we only looked at stationary shock waves $\rho = \rho(x)$ and $T = T(x)$, and now we can look at moving shock waves $\rho = \rho(t, x)$ $T = T(t, x)$



We want to transform to the reference frame of the shock.

- To the right of the shock we have $u_1 = c_s$, $M_1 = \frac{c_s}{a_1}$
- To the left of the shock we have $u_2 = c_s - u_p$,

The Mach number is calculated via the same parameter a used for a standing shock wave.

$$M_1^2 = \frac{\gamma - 1}{2\gamma} + \frac{\gamma + 1}{2\gamma}\frac{p_2}{p_1}$$

$$c_s = M_1 a_1 = a_1 \left(\frac{\gamma - 1}{2\gamma} + \frac{\gamma + 1}{2\gamma} \frac{p_2}{p_1} \right)^{\frac{1}{2}}$$

You can do so similarly with the temperature ratios.

- The pressure, density, and temperature ratios are the same for standing or moving shock waves.
- In the reference frame of the lab, the flow is unsteady
- The flow is steady in the reference frame of the shock

The pressure ratio of a shock wave

$$u_p = \frac{a_1}{\gamma} \left(\frac{p_2}{p_1} - 1 \right) \left(\frac{\frac{2\gamma}{\gamma+1}}{\frac{p_2}{p_1} + \frac{\gamma-1}{\gamma+1}} \right)^{\frac{1}{2}}$$

6.1 Weak Shock Waves

What happens for a weak shock wave, $\frac{\Delta p}{p_1} << 1$. If $\Delta s << 1$, we call this a **nearly isenropic shock**

$$\frac{\rho_2}{\rho_1} = \frac{1 + \frac{\gamma+1}{\gamma-1} \frac{p_2}{p_1}}{\frac{\gamma+1}{\gamma-1} + \frac{p_2}{p_1}}$$

$$1 + \frac{\Delta \rho}{\rho_1} = \frac{1 + \left(1 + \frac{\Delta p}{p_1} \right) \frac{\gamma+1}{\gamma-1}}{\frac{\gamma+1}{\gamma-1} + 1 + \frac{\Delta p}{p_1}}$$

We expand in terms of $\frac{\Delta p}{p_1}$,

$$\frac{\Delta \rho}{\rho_1} = \frac{1}{\gamma} \frac{\Delta p}{p_1} + ...$$

So for a weak shock the ratios above are proportional. We can also relate other quantities for a weak shock:

$$u_p = c_s \left(1 - \frac{u_2}{u_1} \right)$$

$$\frac{u_2}{u_1} = \frac{\rho_1}{\rho_2} = \frac{1}{1 + \frac{\Delta p}{p}}$$

$$u_p = c_s \frac{\Delta \rho}{\rho_1}$$

$$c_s = a_1 \left(\frac{\gamma - 1}{2} + \frac{\gamma + 1}{2\gamma} \frac{p_2}{p_1} \right)^{\frac{1}{2}}$$

$$\frac{u_p}{a_1} = \frac{1}{\gamma} \frac{\Delta p}{p_1}$$

$$\frac{\Delta T}{T_1} = \frac{\gamma - 1}{\gamma} \frac{\Delta p}{p_1}$$

In summary,

- $\Delta \rho$ and ΔT are proportional to Δp
- u_p is small for weak shocks
- As $\Delta p \to 0$ then $c_s \to a_1$.

pg 65 in the book has similar relations for strong shock waves.

6.2 Infinitesimal Waves

- Very small variations in fluid properties
- Perfect gas
- Adiabatic and reversible means isentropic

For isentropic processes, $p = p(\rho)$ and $\frac{p}{\rho^\gamma} = const$. Recall the CE equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

Since

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial x} = a^2\frac{\partial \rho}{\partial x}$$

Since $a^2 = \frac{\partial p}{\partial \rho}|_{s=const}$. So now we have

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{a^2}{\rho}\frac{\partial \rho}{\partial x} = 0$$

For small amplitude disturbances,

$$p_\infty \to p_\infty + p'$$
$$\rho_\infty \to \rho_\infty + \rho'$$
$$u = 0 \to 0 + u'$$
$$a_\infty \to a_\infty + a'$$

Where ∞ refers to the undisturbed conditions and primed quantities indicate the disturbance sizes. We substitute these quantities into the above CE eqns and note that ρ_∞ is constant

$$\frac{\partial \rho'}{\partial t} + u'\frac{\partial \rho'}{\partial x} + (\rho_\infty + \rho')\frac{\partial u'}{\partial x} = 0$$

$$(\rho_\infty + \rho')\frac{\partial u'}{\partial t} + (\rho_\infty + \rho')u'\frac{\partial u'}{\partial x} + (a_\infty + a')^2\frac{\partial \rho'}{\partial x} = 0$$

Now we assume the disturbances are small. That is, we drop the second order terms

$$\frac{\partial \rho'}{\partial t} + \rho_\infty\frac{\partial u'}{\partial x} = 0$$

$$\rho_\infty\frac{\partial u'}{\partial t} + a_\infty^2\frac{\partial \rho'}{\partial x} = 0$$

These are the linearized acoustic equations. If we differentiate, we can convert

$$\frac{\partial^2 \rho'}{\partial t^2} + \rho_\infty\frac{\partial^2 u'}{\partial x \partial t} = 0$$

$$\rho_\infty\frac{\partial^2 u'}{\partial t \partial x} + a_\infty^2\frac{\partial^2 \rho'}{\partial^2 x} = 0$$

into

$$\frac{\partial^2 \rho'}{\partial t^2} - a_\infty^2 \frac{\partial^2 \rho'}{\partial^2 x} = 0$$
$$\frac{\partial^2 u'}{\partial t^2} - a_\infty^2 \frac{\partial^2 u'}{\partial^2 x} = 0$$

Thus we have the solution

$$\rho'(t, x) = F_1(x - a_\infty t) + G_1(x + a_\infty t)$$
$$u'(t, x) = F_2(x - a_\infty t) + G_2(x + a_\infty t)$$

At $t = 0$,

$$\rho'(0, x) = F_1(x) + G_1(x)$$

Each shape propagates at respective characteristic speeds

6.3 Several days of notes during CSE

7 Nonlinear Waves, Finite Disturbances

The nature of the propagation of finite disturbances, etc, depends on the form of the modeling equation.

7.1 Burger's Eqn

$$\partial_t u + u \partial_x u = 0$$

Based on the wave eqn, we expect

$$u(t, x) = f(x - u(t, x)t)$$

So this is an implicit solution depending on the form of the function $f(z)$.

7.2 Nonlinear advection

$$\partial_t u + (a_\infty + u)\partial_x u = 0$$

If we let $a = a(u) = a_\infty + u$, then

$$u(t, x) = f(x - at) = f(x - (a_\infty + u)t)$$

So if $u(0, x) = \sin(x)$, then

$$u(t, x) = \sin(x - (a_\infty + u)t)$$

We notice the characteristics have the property

$$\frac{dx}{dt} = a_\infty + u(t, x)$$

so then at $t = 0$ for the above IC,

$$\frac{dx}{dt} = a_\infty + \sin(x)$$

8 Physical Waves

$$\rho'(x,t) = F(x - a_\infty t) + G(x + a_\infty t).$$

$$u'(x,t) = f(x - a_\infty t) + g(x + a_\infty t).$$

We focus on a single plane wave propagating in the positive x directions. $G = g = 0$.

$$\partial_t \rho' = \partial_t F = -a_\infty F'(x - a_\infty t)$$

$$\partial_x \rho' = \partial_x F = F'(x - a_\infty t)$$

So

$$\partial_t \rho' + a_\infty \partial_x \rho' = 0$$

but similarly

$$\partial_t u' + a_\infty \partial_x u' = 0$$

So $u' = const$ and $\rho' = const$ along $\frac{dx}{dt} = a_\infty$

- Condensation Waves: ρ' is positive and u' could be positive or negative depending on the direction of motion. For a left traveling wave, $\rho' > 0$ and $u' < 0$. For a right traveling wave, $\rho' > 0$ and $u' > 0$
- Rarefactions: Induce fluid motion in opposite direction of wave (expand the gas). For a left traveling wave, $\rho' < 0$ and $u' > 0$. For a right traveling wave, $\rho' < 0$ and $u' < 0$

8.1 Finite Disturbance Waves

Consider the passage of a plane wave through a quiescent gas. We no longer assume small amplitude disturbances. This means nonlinear waves are involved which can alter the sound speed of the medium.

- We assume $p = p(\rho)$ and $u = u(\rho)$ from the Rankine Hugoniot conditions.
- Recall for isentropic waves pressure was proportional to ρ^γ.

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{1 + \frac{\gamma+1}{\gamma-1}\frac{p_2}{p_1}}{\frac{\gamma+1}{\gamma-1} + \frac{p_2}{p_1}}$$

If we plot $\log \frac{\rho_2}{\rho_1}$ vs $\log \frac{p_2}{p_1}$, we have a linear relationship. However, the Rankine Hugoniot conditions deviate from this curve for large pressure differences

We must use the fully nonlinear continuity and momentum equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

Since $u = u(\rho)$ and $p = p(\rho)$ (velocity and pressure depend only on density)

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial u}\frac{\partial u}{\partial t} \qquad \frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial u}\frac{\partial u}{\partial x} \qquad \frac{\partial p}{\partial x} = \frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial u}\frac{\partial u}{\partial x}$$

We substitute this into the continuity equations

$$\frac{\partial \rho}{\partial u}\frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial u}\frac{\partial u}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial u}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)+\rho\frac{\partial u}{\partial x}=0$$

We do the same for the momentum equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\frac{-1}{\rho}\frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial u}\frac{\partial u}{\partial x}$$

Then we notice we can use this in the continuity equation to obtain

$$\frac{\partial \rho}{\partial u}\left(\frac{-1}{\rho}\frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial u}\frac{\partial u}{\partial x}\right)+\rho\frac{\partial u}{\partial x}=0$$

$$\left(\frac{\partial \rho}{\partial u}\right)^2\frac{1}{\rho^2}\frac{\partial p}{\partial \rho}\frac{\partial u}{\partial x}=\frac{\partial u}{\partial x}$$

Which gives us

$$\partial u=\pm\sqrt{\frac{\partial p}{\partial \rho}}\frac{\partial \rho}{\rho}$$

Recall $a^2=\frac{\partial p}{\partial \rho}$ at constant entropy. However, since for these disturbances pressure is a function of density only, this is a proper derivative $a^2=\frac{dp}{d\rho}$. This a is now a local quantity and this relationship holds in isentropic regimes (away from shocks).

$$\frac{du}{a}=\pm\frac{d\rho}{\rho}$$

This is analogous to the equations for infinitesimal disturbances.

- The positive sign is for a forward traveling wave, the negative sign is for a backward traveling wave.
- The flow velocity follows a compression wave
- The flow velocity moves away from an expansion/rarefraction wave

Now consider only the forward traveling wave

$$\frac{du}{a}=\frac{d\rho}{\rho}$$

Combine this with the momentum equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\frac{-1}{\rho}a^2\frac{\rho}{a}\frac{\partial u}{\partial x}=-a\frac{\partial u}{\partial x}$$

or in terms of the nonlinear scalar advection equation

$$\frac{\partial u}{\partial t}+(u+a)\frac{\partial u}{\partial x}=0$$

with solution

$$u(x,t)=f(x-(u(x,t)+a)t)$$

Note that a is the local sound speed which varies inside the wave. By definition a is the sound speed for isentropic infinitesimal disturbances, so we use

$$\frac{p}{\rho^\gamma}=\frac{p_\infty}{\rho_\infty^\gamma}\implies\frac{dp}{d\rho}=\frac{p_\infty}{\rho_\infty^\gamma}\gamma\rho^{\gamma-1}$$

So then

$$a = \sqrt{\frac{p_\infty}{\rho_\infty^\gamma}\gamma\rho^{\gamma-1}} = \sqrt{\frac{\gamma p_\infty}{\rho_\infty}\frac{\rho^{\gamma-1}}{\rho_\infty^{\gamma-1}}} = a_\infty\left(\frac{\rho}{\rho_\infty}\right)^{\frac{\gamma-1}{2}}$$

This is for a forward traveling wave. Thus

$$du = a\frac{d\rho}{\rho} = \frac{a_\infty}{\rho^{\frac{\gamma-1}{2}}}\rho^{\frac{\gamma-3}{2}}d\rho$$

Integrate

$$u = \frac{a_\infty}{\rho^{\frac{\gamma-1}{2}}}\frac{\rho^{\frac{\gamma-1}{2}}}{\frac{\gamma-1}{2}} + const$$

Then we use $u = 0 \implies \rho = \rho_\infty$, which is the same as $u' = \rho' = 0$. Thus

$$const = \frac{-2a_\infty}{\gamma - 1}$$

So then

$$u = \frac{2}{\gamma-1}(a - a_\infty)$$

$$a = a_\infty + \frac{\gamma-1}{2}u$$

- $a > a_\infty$ for $u > 0$ and vice versa
- $a - a_\infty$ is proportional to u
- The disturbance travels at speed $c = u + a = a_\infty + u + \frac{\gamma-1}{2}u = a_\infty + \frac{\gamma+1}{2}u$
- The wave travels at a speed greater than the sound speed for $u > 0$ and slower when $u < 0$.
- The wave is no longer travels at a speed that is independent of u. It is a function of the local fluid velocity.
- For infinitesimal waves

  $$\frac{u'}{a_\infty} = \frac{\rho'}{\rho_\infty} << 1, u \to u'$$

  In the limit $\frac{u'}{a_\infty} << 1$, we recover $c \to a_\infty$. Since $c = c(u)$,waves don't always propagate at a steady speed.
- Characteristics converge to form a shock wave
- Irreversible process since information that propagates into the sock is destroyed (uniqueness is lose)
- Shock waves are called the black hole of characteristics since they absorb all signals and information
- The entropy is discontinuous at shocks
- If characteristics do not converge flow can be reverse without loss of information

8.2 Steady Shock Wave Propagation

- Finite disturbances show propagation velocity increases with wave amplitude

$$c = u + a = a_\infty + \frac{\gamma + 1}{2} u$$

How does a steady shock exist? The shock separates two steady states or constant states. Consider the Riemann problem with $u = u_p$ (piston velocity) for $x < 1$ and $u = 0$ for $x > 1$. Thus for $x < 1$ the characteristics have the form $x = (a_\infty + \frac{\gamma+1}{2} u_p)t + x_0$. For $x > 1$ $x = a_\infty t + x_0$. At $x = 1$, $x = c_s t + x_0$ where c_s is the sound speed. Thus on either side we have parallel characteristics, but since $a_\infty + \frac{\gamma+1}{2} u_p > c_s > a_\infty$, they converge. This process sustains a shack wave at a constant speed of propagation.

9 Simple Centered Expansion Waves

What if instead of pushing with a piston we pull? The fluid must fill the void.

- We assume the flow is isentropic since this process is reversible and we have added no heat.

We assume the piston moves instantaneously. This gives rise to a centered expansion fan. Centered refers to the fact that the fan arises from a single point.

- The piston characteristic: $x_p = -|u_p| t$
- The fan head characteristic: $x_4 = c_4 t = \left(a_\infty + \frac{\gamma+1}{2} u\right) t$ with $a_\infty = a_4$ and $u = u_4 = 0$, so then

$$x_4 = a_4 t$$

- The fan tail characteristic: $x_3 = c_3 t = \left(a_\infty + \frac{\gamma+1}{2} u\right) t$ with $a_\infty = a_4$ and $u = u_3 = -|u_p|$ so

$$x_3 = \left(a_4 - \frac{\gamma + 1}{2} |u_p|\right) t$$

If $c = \frac{dx}{dt} = const$ along a characteristic (the characteristics are straight lines), we have a simple centered fan. We integrate to obtain $c = \frac{x}{t}$ since $x_0 = t_0 = 0$. So then for the tail of our expansion,

$$c_3 = a_4 - \frac{\gamma + 1}{2} |u_p| = const$$

  - The velocity inside of region 3 downstream of the tail is constant $u_3 = -|u_p|$.
  - Velocity upstream of expansion in region 4 is constant $u_4 = 0$
  - What about inside the expansion?

$$c = \frac{x}{t} = a_4 + \frac{\gamma + 1}{2} u \implies u = \frac{2}{\gamma + 1} \left(\frac{x}{t} - a_4\right)$$

  - Velocity varies linearly with position
  - The slope at a fixed point is inversely proportional to time.

We want to relate u to the other flow variables. Recall

$$c = \left(a_4 + \frac{\gamma + 1}{2} u\right) t$$

which was derived from forward propagating finite disturbance.

$$du = a\frac{d\rho}{\rho} \qquad \text{or for backwards moving waves } du = -a\frac{d\rho}{\rho}$$

Here $a_\infty = a_4$

$$1 = \frac{a}{a_4} - \frac{\gamma - 1}{2}\frac{u}{a_4} \implies \frac{a}{a_4} = 1 + \frac{\gamma - 1}{2}\frac{u}{a_4} \implies a < a_4$$

Use $a = a\sqrt{\gamma RT}$,

$$\frac{T}{T_4} = \left(1 + \frac{\gamma - 1}{2}\frac{u}{a_4}\right)^2$$

- Expansion waves are adiabatic
- An expansion fan is a reversible adiabatic (isentropic) process since the characteristic diverge
- We can use the isentropic relations

$$\frac{p}{p_4} = \left(\frac{\rho}{\rho_4}\right)^\gamma = \left(\frac{T}{T_4}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{p}{p_4} = \left(1 + \frac{\gamma - 1}{2}\frac{u}{a_4}\right)^{\frac{2\gamma}{\gamma-1}} \implies p < p_4$$

$$\frac{\rho}{\rho_4} = \left(1 + \frac{\gamma - 1}{2}\frac{u}{a_4}\right)^{\frac{2}{\gamma-1}} \implies \rho < \rho_4$$

10 Entropy

We can include energy to obtain a third characteristic

$$c_0 = u$$

This is sometimes called an entropy wave

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} = 0$$

- There can be an entropy discontinuity that advects at local fluid velocity.
- Contact waves/ discontinuities occur when speed u and pressure are continuous while other properties jump.
- Across a contact all characteristics are parallel

  $$\rho(x,t) = f(x - ut)$$

  Velocity u is equal on either side of the wave. Behaves like a linear advection equation.
- If we apply conservation of mass in the wave reference frame, we have zero mass flux across the wave.

  If $p = const$ we can use the ideal gas law

$$\rho_3 R_3 T_3 = \rho_2 R_2 T_2$$

We have different ways to have a discontinuity

- Different gases same temperature: $\rho_3 R_3 = \rho_2 R_2$, $R_i = \frac{R}{w_i}$
- Same gases different temperature: $\rho_3 T_3 = \rho_2 T_2$
- Different gases different temperature

Note:

- Entropy may change across a contact
- Contact discontinuities do not create entropy but they may move a jump in the entropy
- Second law of thermodynamics says entropy must increase following a fluid particle. There is no fluid that crosses a contact, thus there is no violation of the second law.
- Contacts are shear layers or slip lines in multiple dimensions.

11 Riemann Problems

Imagine state 1 and state 4 separated by a diaphragm which vanishes at $t = 0$. Before this $u_1 = u_4 = 0$, and $p_1 < p_4$. This creates a shock, a contact, and an expansion wave.

- Between 1 and 2 we have a shock propagating at velocity $-c_s$
- between 2 and 3 we have a contact
- Between 3 and 4 we have an Expansion fan, the head of which is propagating at velocity a_4

Given $\frac{p_4}{p_1}$, we can find the solution inside the four regions using relations for shocks, contacts, and expansions. That is, we know

$$(\rho, u, p)|_{\text{region 1}} = (\rho_1, 0, p_1)$$
$$(\rho, u, p)|_{\text{region 4}} = (\rho_4, 0, p_4)$$

Across the shock,

$$u_2 = u_p = \frac{a_1}{\gamma}\left(\frac{p_2}{p_1} - 1\right)\left(\frac{\frac{2\gamma}{\gamma+1}}{\frac{p_2}{p_1} + \frac{\gamma-1}{\gamma+1}}\right)^{\frac{1}{2}} = \frac{a_1}{\gamma}\left(\frac{p_2}{p_1} - 1\right)\left(\frac{2\gamma}{(\gamma+1)\frac{p_2}{p_1} + \gamma - 1}\right)^{\frac{1}{2}} \tag{1}$$

Across the expansion wave,

$$\frac{p_3}{p_4} = \left(1 - \frac{\gamma-1}{2}\frac{u_3}{a_4}\right)^{\frac{2\gamma}{\gamma-1}}$$
$$u_3 = \frac{2a_4}{\gamma-1}\left(1 - \left(\frac{p_3}{p_4}\right)^{\frac{\gamma-1}{2\gamma}}\right) \tag{2}$$

Across the contact

$$u_2 = u_3$$
$$p_2 = p_3$$

Which means we can equate 1 and 2.

$$\frac{a_1}{\gamma}\left(\frac{p_2}{p_1} - 1\right)\left(\frac{2\gamma}{(\gamma+1)\frac{p_2}{p_1} + \gamma - 1}\right)^{\frac{1}{2}} = \frac{2a_4}{\gamma-1}\left(1 - \left(\frac{p_3}{p_4}\right)^{\frac{\gamma-1}{2\gamma}}\right)$$

Recall that $p_1 = const p_4$ and that $\frac{p_3}{p_4} = \frac{p_2}{p_4} = \frac{p_2}{p_1}\frac{p_1}{p_4}$

$$\frac{p_4}{p_1} = \frac{p_2}{p_1}\left(1 - \frac{\gamma-1}{2}\frac{a_1}{a_4}\left(\frac{p_2}{p_1} - 1\right)\sqrt{\frac{\frac{2}{\gamma}}{(\gamma+1)\frac{p_2}{p_1} + \gamma - 1}}\right)^{\frac{-2\gamma}{\gamma-1}}$$

We know $\frac{p_4}{p_1}$ and $\frac{a_1}{a_4}$. The last remaining unknown is $\frac{p_2}{p_1}$. Once this is obtained (numerically), we know the shock jump conditions. We can find p_3 then $u_p = u_2 = u_3$. Now the only unknown is the velocity inside the expansion

$$c = \frac{x}{t} = a_4 + \frac{\gamma+1}{2}u$$

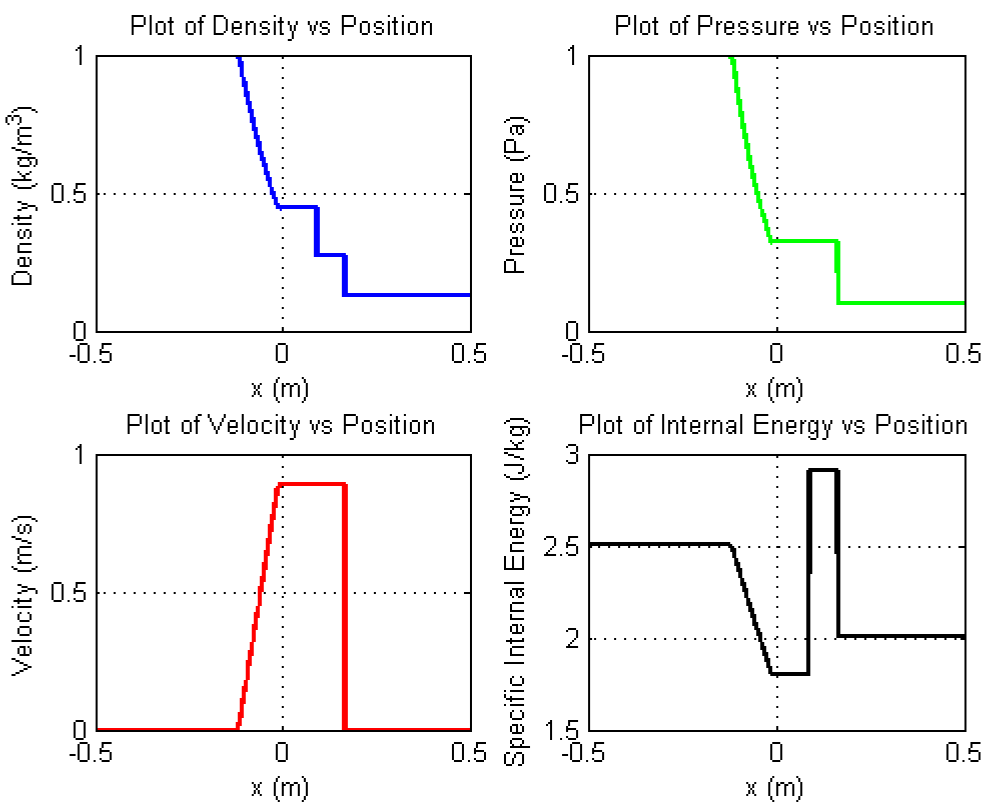
or

$$u = \frac{2}{\gamma+1}\left(\frac{x}{t} - a_4\right)$$

So then

$$u(t,x) = \begin{cases} 0 & x < -c_s t \\ -|u_p| & -c_s t < x < -|u_p|\, t \\ ?? & ?? \\ 0 & a_4 t < x \end{cases}$$

Insert a picture of compressible Euler shock tube experiments



12 Wave Reflections

There are two types of reflections

- Reflections off of a wall, BC $u = 0$. The shock wave will reflect off the wall.
- or reflections off the opening boundary (like a tube open at one end.

Part III
2D Flow

13 2D Supersonic Flow

- Oblique shocks
- Flow over wedges
- Mach lines
- Weak Oblique shocks
- Supersonic compression
- Supersonic expansions
- Reflected shocks
- Detached shocks

13.1 Oblique Shocks

See drawing in notes. We decompose the incoming velocity into each dimension then apply the compressible euler equations.
Across the shock $u_2 < u_1$ but $v_1 = v_2 = v$. The flow turns towards the shock.