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Part I
Finite Element Methods

Notation. Let the default norm be the $L^2(a,b)$ norm. We define the weighted a**-norm** and the **Energy norm** as such.

$$\begin{aligned}\|\psi\| &= \|\psi\|_{L^2(a,b)}\\ \|\psi\|_a &= \left\|\sqrt{a}\psi\right\|_{L^2(a,b)} = \int_a^b a(x)(\psi(x))^2dx\\ \|\psi\|_E &= \|\|\end{aligned}$$

1 Introduction

Consider the BVP

$$\begin{cases} -\left(a(x)u'\right)' = f(x) & x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}, 0 < a_{min} < a(x) < a_{max} < \infty, \qquad (\textbf{BVP1})$$

Notice BVP1 is self adjoint with $0 \notin \sigma_p(L)$. This has a solution as long as f is continuous and $a \in C^1$.
We reformulate this as the variational problem

$$\int_0^1 au'v'dx = \int_0^1 fvdx, v \in \mathcal{H}_0^1 \qquad (\textbf{VAR1})$$

which has a unique solution.

2 cG(1): Galerkin FEM

The continous Glaerkin Method of Order 1 considers the variational problem and chooses hat functions as the trial and test functions .

2.1 Neumann BCs

Consider

$$\begin{cases} -\left(a(x)u'\right)' = f(x) & x \in (0,1) \\ u(0) = 0; a(1)u'(1) = g_1 \end{cases}, 0 < a_{min} < a(x) < a_{max} < \infty, \qquad (\textbf{BVPN})$$

We choose $V = \left\{v \in \mathcal{H}_0^1, v(0) = 0\right\}$ so the variational problem is

$$\int_0^1 au'v'dx - g_1v(1) = \int_0^1 fvdx$$

We then use the approximate test fuction space

$$V_h^1 = \{\phi_1, ..., \phi_M, \phi_{M+1}\}$$

Where ϕ_{M+1} is an added half-hat function. Let $U(x) = \sum\limits_{j=1}^{M+1} \xi_j\phi_j(x)$ so our variational problem becomes

$$\sum_{j=1}^{M+1} \xi_i \int_0^1 a(x)\phi_j'(x)\phi_i'(x)dx - g_1v(1) = \int_0^1 f\phi_idx + g_1\phi_i(1)$$

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3 Ritz FEM

4 New Notes

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Note that for ψ st $-(a\psi')' = \xi$, $\psi(0) = \psi(1) = 0$,

$$S_a = \max_{\xi \neq 0} \frac{\|\psi''\|_a}{\xi} \leq C(a)$$

So

$$\|a\psi''\| \leq \|f\| + \|a'\psi'\| \leq \|f\| + \|a'\|_\infty \|\psi'\|$$

4.1 An a posteriori error estimate

Let $e = u - U$, then

$$\|e\|_E^2 = \int_0^1 ae'e'dx = \int_0^1 ae'u' - \int_0^1 ae'U'dx = \int_0^1 fedx - \int_0^1 ae'U'dx$$

Subtract from this

$$\int_0^1 fvdx - \int_0^1 aU'v'dx = 0$$

So we have

$$\begin{aligned} \|e\|_E^2 &= \int_0^1 f(e - P_n e)dx - \int_0^1 aU'(e - P_n e)'dx \\ &= \sum_{j=1}^{M+1} \int_{x_{j-1}}^{x_j} \left(f(e - P_n e) - aU'(e - P_n e)'\right) dx \\ &= \sum_{j=1}^{M+1} \int_{x_{j-1}}^{x_j} \left(f + (aU')'\right)(e - P_n e)\, dx \sum_{j=1}^{M+1} \left[-aU'(e - P_n e)\right]_{x_{j-1}}^{x_j} \end{aligned}$$

Since $(e - P_n e)|_{x_j} = 0$ for all $j = 0, 1, ..., M+1$

$$\|e\|_E^2 = \sum_{j=1}^{M+1} \int_{x_{j-1}}^{x_j} R(U)\,(e - P_n e)\, dx = \int_0^1 R(U)\,(e - P_n e)\, dx$$

where $R(U) = f + (aU')'$.

Cauchy,:

$$\|e\|_E^2 \leq \|hR(u)\|_{a^{-1}} \left\|h^{-1}(e - P_n e)\right\|_a$$

Note that $\left\|h^{-1}(e - P_n e)\right\|_a \leq C\|E\|_a$.

Theorem: Error Estimate There exists a constant C depending only on $a(x)$ such that the cG(1) FEM approximation U satisfies

$$\|u - U\|_E = \|(u - U)'\|_a \leq C(a)\|hR(U)\|_{a^{-1}}$$

Adaptive Mesh Refinement Steps

1. Compute the approximate FEM solution on a uniform coarse mesh.

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2. Calculate the error on each element K_j
$$E_j^2 = (x_j - x_{j-1})^2 \int_{x_{j-1}}^{x_j} \frac{1}{a(x)} R^2(U)dx$$
If $E_j^2 \leq \frac{\epsilon^2}{c^2(M+1)}$, move onto next element.
If $E_j^2 > \frac{\epsilon^2}{c^2(M+1)}$, subdivide the element into 2 or more elements, then move onto the next element.
3. Recalculate solution on the new mesh.
4. Repeat 2-3 until no subdivision is required.

Note that the constant c is unknown. The more accurately we know this c, the better our estimate will be and therefore the more efficient our code will be. Estimates for c can be given by calibration based on known solutions.

4.1.1 L_2 norm a-posteriori error estimate

$$\|u - U\| \leq C_s C_I \left\|h^2 R(U)\right\|$$

where
$$\left\|h^{-1}(\phi - P_n\phi)\right\| \leq C_I \|\phi''\|$$

and
$$C_s = \max_{\xi \neq 0} \frac{\|\psi''\|}{\xi}, \qquad -(a\psi')' = \xi$$

Proof. We have the dual problem $-(a\psi')' = e, \phi(0) = \phi(1) = 0$, so

$$\begin{aligned}
\|e\|^2 &= -\int_0^1 e(a\psi')'dx = \int_0^1 (u-U)(a\phi')'dx \\
&= \int_0^1 a(u-U)'\phi'dx - [(u-U)a\phi']_0^1 \\
&= \int_0^1 au'\phi'dx - \int_0^1 aU'\phi'dx \\
&= \int_0^1 f\phi dx - \int_0^1 aU'\phi'dx - \left(\int_0^1 fvdx - \int_0^1 aU'v'dx\right), v = P_n\phi \\
&= \int_0^1 \left(f(\phi - P_n\phi) - \int_0^1 aU'(\phi - P_n\phi)'\right) dx \\
&= \sum_{i=0}^{M+1} \int_{x_{j-1}}^{x_j} R(U)(\phi - P_n\phi)dx \\
&= \int_0^1 R(U)(\phi - P_n\phi)dx \\
&\leq \left\|h^2 R(U)\right\| \left\|h^{-2}(\phi - P_n\phi)\right\| \\
&\leq \left\|h^2 R(U)\right\| C_I\|\phi''\| \\
&\leq \left\|h^2 R(U)\right\| C_I C_S \|e\| \\
\|e\| &\leq \left\|h^2 R(U)\right\| C_I C_S
\end{aligned}$$

Using this, we modify step 2 of the adaptive mesh refinement

$$E_j^2 = (x_j - x_{j-1})^4 \int_{x_{j-1}}^{x_j} R^2(U(x))dx$$

and let $c = C_I C_s$ in step 2.

5 cG(2) Method

Consider our favorite BVP/VAR

$$-(au')' = f, u(0) = u(1) = 0 \iff \int_0^1 au'v'dx = \int_0^1 fvdx, \; \forall\, v \in V = \mathcal{H}_0^1$$

Now we use the approximate V_h^2 :

$$V_h^2 = \left\{ v \in V : v|_{k_j} = a_j + b_j x + c_j x^2 \right\}$$

We seek a basis $\{\phi_n\}$ such that $V_h^2 = \text{span}\,\{\phi_n\}$.

Given a mesh $T_h = \{h_i\}_{M+1}$ so that $x_j = x_0 + \sum\limits_{i=1}^{j} h_i$, we introduce a submesh that divides the mesh T_h into subintervals of size $\frac{1}{2}$. We then consider the Lagrange polynomials

$$(1,0,0) \coloneqq p_1(x_{j-1}) = 1, p_1(x_{j-\frac{1}{2}})_1 = p(x_j) = 0$$

$$(0,1,0) \coloneqq p_2(x_{j-\frac{1}{2}}) = 1, p_2(x_{j-\frac{1}{2}})_1 = p_2(x_j) = 0$$

$$(0,0,1) \coloneqq p_3(x_j) = 1, p_3(x_{j-\frac{1}{2}})_1 = p_3(x_{j-1}) = 0$$

$$\phi_i(x) = \begin{cases} \frac{2(x-x_{i+\frac{1}{2}})(x-x_{i+1})}{(x_{i+1}-x_i)^2} & x_i \le x \le x_{i+1} \\ \frac{2(x-x_{i-\frac{1}{2}})(x-x_{i-1})}{(x_i-x_{i-1})^2} & x_{i-1} \le x \le x_i \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_{i-\frac{1}{2}}(x) = \begin{cases} \frac{4(x_i-x)(x-x_{i-1})}{(x_i-x_{i-1})^2} & x_{i-1} \le x \le x_i \\ 0 & \text{otherwise} \end{cases}$$

$$V_h^2 = \text{span}\left\{ \phi_{\frac{1}{2}}, \phi_1, \phi_{\frac{3}{2}}, \phi_2, ..., \phi_M, \phi_{M+\frac{1}{2}} \right\}$$

$$U(x) = \sum_{j=1}^{2M+1} \xi_{\frac{j}{2}} \phi_{\frac{j}{2}}(x)$$

So for the same mesh T_h, the cG(2) has roughly twice the amount of unknowns as cG(1) method.

For any function $v \in V$,

$$(P_h v)x) = \sum_{j=1}^{2M+1} v(x_{\frac{j}{2}}) \phi_{\frac{j}{2}}(x)$$

cG(2) variational problem

$$\sum_{j=1}^{2M+1} \xi_{\frac{j}{2}} \int_0^1 a(x) \phi'_{\frac{j}{2}}(x) \phi_{\frac{i}{2}}(x)dx = \int_0^1 f(x)\phi_{\frac{i}{2}}(x)dx$$

for $i = 1, 2, ..., 2M + 1$.

5.0.2 Example

Consider $a \equiv 1$ and $h_j = h, j = 1, ..., M+1$.
Stiffness matrix
If i is odd, $i = 1, 3, ..., 2M+1$,

$$a_{ii} = \int_{x_{\frac{i-1}{2}}}^{x_{\frac{i+1}{2}}} (\phi'_{\frac{i}{2}}(x))^2 dx = \frac{16}{3h}$$

$$a_{i-1,i} = a_{i,i-1} = \int_{x_{\frac{i-1}{2}}}^{x_{\frac{i+1}{2}}} \phi'_{\frac{i}{2}}(x)\phi'_{\frac{i-1}{2}}(x)dx = \frac{-8}{3h}$$

If i even, $i = 2, 4, ..., 2M$,

$$a_{ii} = \int_{x_{\frac{i-1}{2}}}^{x_{\frac{i+1}{2}}} (\phi'_{\frac{i}{2}}(x))^2 dx = \frac{14}{3h}$$

$$a_{i-1,i} = a_{i,i-1} = \int_{x_{\frac{i-1}{2}}}^{x_{\frac{i+1}{2}}} \phi'_{\frac{i}{2}}(x)\phi'_{\frac{i-1}{2}}(x)dx = \frac{-8}{3h}$$

$$a_{i-2,i} = a_{i,i-2} = \int_{x_{\frac{i-1}{2}}}^{x_{\frac{i}{2}}} \phi'_{\frac{i}{2}}(x)\phi'_{\frac{i-2}{2}}(x)dx = \frac{1}{3h}$$

You can check this because summing the rows gives a zero vector.

$$f = P_h f = \sum_{j=1}^{2M+1} f(x_{\frac{j}{2}})\phi_{\frac{j}{2}}(x) = \sum_{j=1,odd}^{2M+1} f(x_{\frac{j}{2}})\phi_{\frac{j}{2}}(x) + \sum_{j=2,even}^{2M} f(x_{\frac{j}{2}})\phi_{\frac{j}{2}}(x)$$

Odd pieces: For odd i,

$$\begin{aligned} f_i = \sum_{j=1,odd}^{2M+1} f(x_{\frac{j}{2}}) \int_{x_{\frac{i-1}{2}}}^{x_{\frac{i+1}{2}}} \phi_{\frac{j}{2}}(x)\phi_{\frac{i}{2}}(x) =& f(x_{\frac{i-1}{2}}) \int_{x_{\frac{i-1}{2}}}^{x_{\frac{i+1}{2}}} \phi_{\frac{i-1}{2}}(x)\phi_{\frac{i}{2}}(x)dx \\ &+ f(x_{\frac{i}{2}}) \int_{x_{\frac{i-1}{2}}}^{x_{\frac{i+1}{2}}} (\phi_{\frac{i}{2}}(x))^2 dx \\ &+ f(x_{\frac{i+1}{2}}) \int_{x_{\frac{i-1}{2}}}^{x_{\frac{i+1}{2}}} \phi_{\frac{i+1}{2}}(x)\phi_{\frac{i}{2}}(x)dx \end{aligned}$$

so

$$f_i = \frac{h}{15}\left(f(x_{\frac{i-1}{2}}) + 8f(x_{\frac{i}{2}}) + f(x_{\frac{i+1}{2}})\right)$$

Even pieces: For even i,

$$f_i = \frac{h}{30}\left(-f(x_{\frac{i-2}{2}}) + 2f(x_{\frac{i-1}{2}}) + 8f(x_{\frac{i}{2}}) + 2f(x_{\frac{i+1}{2}}) - f(x_{\frac{i+2}{2}})\right)$$

This produces a matrix problem $Au = f$, where A is SPD.

5.0.3 Error Analysis

Energy norm error

$$\|u - U\|_E \leq \|u - v\|_E, \ \forall \ v \in V_h^2$$

Take $v = P_h u$, then

$$\|u - U\|_E \leq \|u - P_h u\|_E \leq ch^2\|u'''\|$$

L_2 norm error:

$$\|u - U\|_{L^2} \leq ch^3\|u'''\|$$

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6 Beam Equation

$$\begin{cases} u^{(4)} = f & 0 < x < 1 \\ u(0) = u(1) = u'(0) = u'(1) = 0 \end{cases} \text{(BEAM BVP)}$$

u and u' are specified at the boundary, thus are essential boundary conditions. v and v' will be strongly enforce. So let

$$V = \left\{ v \in H^2 : v(0) = v(1) = v'(0) = v'(1) = 0 \right\}$$

So we have our Variational Problem

$$\int_0^1 vu''''dx = \int_0^1 vfdx \iff \int_0^1 u''v''dx = \int_0^1 fvdx \text{(BEAM VAR)}$$

Recasting this as an energy minimization, we define the functional

$$F[w] = \frac{1}{2}\int_0^1 (w'')^2 dx - \int_0^1 fwdx$$

So we seek $u \in V$ such that

$$F[u] \leq F[v], \qquad \forall\, v \in V \text{(BEAM MIN)}$$

Assume $w = u + v$, $v \in V$.

$$F[w] = F[u+v] = F[u] + \frac{1}{2}\int_0^1 (v'')^2 dx + \int_0^1 u''v''dx - \int_0^1 fvdx = F[u] + \frac{1}{2}\int_0^1 (v'')^2 dx \geq F[u]$$

So minimum energy is achieved with $\int_0^1 (v'')^2 dx = 0$ which with the boundary conditions implies $v(x) = 0$.

6.1 cG(3) Method

The regularity condition of our V for our Beam Variational problem is continuously differentiable functions. Thus we need cubics in order to match the values and derivatives of neighboring solutions (4 conditions implies 4 coefficients).

$$V_h^3 = \left\{ v \in V : v|_{kj} = a_j + b_j x + c_j x^2 + d_j x^3 \right\}$$

So our first basis function is (recall $h_j = x_j - x_{j-1}$)

$$\phi_j^{(1)}(x) = \begin{cases} 3h_j^{-2}(x - x_{j-1})^2 - 2h_j^{-3}(x - x_{j-1})^3 & x_{j-1} \leq x \leq x_j \\ 3h_{j+1}^{-2}(x - x_{j+1})^2 + 2h_{j+1}^{-3}(x - x_{j+1})^3 & x_j \leq x \leq x_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

And we note

$$(\phi_j^{(1)}(x))' = \begin{cases} 6h_j^{-2}(x - x_{j-1}) - 6h_j^{-3}(x - x_{j-1})^2 & x_{j-1} \leq x \leq x_j \\ 6h_{j+1}^{-2}(x - x_{j-1}) + 6h_{j+1}^{-3}(x - x_{j+1})^2 & x_j \leq x \leq x_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

Note that

- $\phi_j^{(1)}(x) \in C^1[0,1]$, $\phi_j^{(1)}(x) \notin C^2[0,1]$
- $\phi_j^{(1)}(x_k) = \delta_{jk}$, $(\phi_j^{(1)})'(x_k) = 0$

Our second basis function is

$$\phi_j^{(2)}(x) = \begin{cases} -h_j^{-1}(x-x_{j-1})^2 + h_j^{-2}(x-x_{j-1})^3 & x_{j-1} \le x \le x_j \\ h_{j+1}^{-1}(x-x_{j+1})^2 + h_{j+1}^{-2}(x-x_{j+1})^3 & x_j \le x \le x_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

And we note

$$(\phi_j^{(2)}(x))' = \begin{cases} -2h_j^{-1}(x-x_{j-1}) + 3h_j^{-2}(x-x_{j-1})^2 & x_{j-1} \le x \le x_j \\ 2h_{j+1}^{-1}(x-x_{j+1}) + 3h_{j+1}^{-2}(x-x_{j+1})^2 & x_j \le x \le x_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

Note that

- $\phi_j^{(2)}(x) \in C^1[0,1]$, $\phi_j^{(2)}(x) \notin C^2[0,1]$
- $\phi_j^{(2)}(x_k) = 0$, $(\phi_j^{(2)})'(x_k) = \delta_{kj}$

So we have

$$V_h^3 = \operatorname{span}\left\{\phi_1^{(1)}, ..., \phi_M^{(1)}, \phi_1^{(2)}, ..., \phi_M^{(2)}\right\}$$

So we express our trial function

$$U = \sum_{j=1}^{M} \xi_j \phi_j^{(1)}(x) + \sum_{j=1}^{M} \eta_j \phi_j^{(2)}(x)$$

Note

- $U(x_k) = \xi_k$, $U'(x_k) = \eta_k$

So for any $v \in V$,

$$P_h v = \sum_{j=1}^{M} v(x_j)\phi_j^{(1)}(x) + \sum_{j=1}^{M} v'(x_j)\phi_j^{(2)}(x)$$

Plugging this into our variational problem,

$$\sum_{j=i-1}^{i+1} \xi_j \int_{x_{i-1}}^{x_{i+1}} (\phi_i^{(l)})''(\phi_j^{(1)})'' dx + \sum_{j=i-1}^{i+1} \eta_j \int_{x_{i-1}}^{x_{i+1}} (\phi_i^{(l)})''(\phi_j^{(2)})'' dx = RHS$$

This results in a matrix equation

$$S\vec{\mathbf{u}}^* = \begin{pmatrix} A & C \\ -C & B \end{pmatrix} \begin{pmatrix} \vec{\xi} \\ \vec{\eta} \end{pmatrix} = \begin{pmatrix} \vec{\mathbf{f}}^{(1)} \\ \vec{\mathbf{f}}^{(2)} \end{pmatrix}$$

A is tridiagonal, symmetric:

$$A = \frac{12}{h^3} \begin{pmatrix} 2 & -1 & \\ -1 & 2 & 1 \\ \ddots & \ddots & \ddots \end{pmatrix}_{M \times M}$$

B is tridiagonal, symmetric:

$$B = \frac{2}{h} \begin{pmatrix} 4 & 1 & \\ 1 & 4 & 1 \\ \ddots & \ddots & \ddots \end{pmatrix}_{M \times M}$$

C is tridiagonal, antisymmetric defined by $c_{ij} = \int_{x_{i-1}}^{x_{i+1}} (\phi_i^{(1)})''(\phi_j^{(2)})''dx$ so it looks like

$$C = \frac{6}{h^2}\begin{pmatrix} 0 & 1 & \\ -1 & 0 & 1 \\ \ddots & \ddots & \ddots \end{pmatrix}_{M\times M}$$

So $S_{2M\times 2M}$ is symmetric, positive definite thus invertible. For the right hand side of the equation, we project f onto our test functions

$$P_v f = \sum_{j=0}^{M+1} f(x_j)\phi_j^{(1)}(x) + \sum_{j=0}^{M+1} f'(x_j)\phi_j^{(2)}(x)$$

So then the RHS of the equation

$$RHS = \sum_{k=i-1}^{i+1}\left(f(x_k)\int_{x_{i-1}}^{x_{i+1}} \phi_i^{(l)}\phi_k^{(l)}dx + f'(x_k)\int_{x_{i-1}}^{x_{i+1}} \phi_i^{(l)}\phi_k^{(2)}dx\right)$$

Using quadrature,

$$f_i^{(1)} = \frac{h}{70}\left(9f(x_{i-1}) + 52f(x_i) + 9f(x_{i+1})\right) + \frac{13h^2}{420}\left(f'(x_{i-1}) - f'(x_{i+1})\right)$$

$$f_i^{(2)} = \frac{13h^2}{420}\left(f(x_{i+1}) - f'(x_{i-1})\right) - \frac{h^3}{420}\left(3f'(x_{i-1}) - 8f'(x_i) + 3f'(x_{i+1})\right)$$

6.1.1 Error Analysis

We know our interpolation error goes as

$$\|u - P_h u\|_{L^2(0,1)} \le C_1 h^4 \left\|u^{(4)}\right\|_{L^2}$$

$$\|(u - P_h u)'\|_{L^2(0,1)} \le C_2 h^3 \left\|u^{(4)}\right\|_{L^2}$$

$$\|(u - P_h u)''\|_{L^2(0,1)} \le C_3 h^2 \left\|u^{(4)}\right\|_{L^2}$$

A priori error estimate (Energy Norm) Exact variational problem is finding $u \in V$ such that

$$\int_0^1 u''v''dx = \int_0^1 fvdx \;\forall\; v \in V$$

our approximate problem is finding $U'' \in V_h^3$

$$\int_0^1 U''v''dx = \int_0^1 fvdx \;\forall\; v \in V$$

Subtracting, we get the Galerkin orthogonality condition

$$\int_0^1 (u-U)''v''dx = 0 \;\forall\; v \in V_h^3$$

So then we see

$$\|u-U\|_E^2 = \int_0^1 ((u-U)'')^2dx = \int_0^1 (u-U)''(u-v+v-U)''dx$$

Galerkin orthogonality tells us that this becomes

$$\|u-U\|_E^2 = \int_0^1 (u-U)''(u-v)''dx + \int_0^1 (u-U)''(v-U)''dx = \int_0^1 (u-U)''(u-v)''dx \;\forall\; v \in V_h^3$$

So then

$$\|u-U\|_E^2 \le \|u-U\|_E\|u-v\|_E \implies \|u-U\|_E \le \|u-v\|_E \;\forall\; V_h^3$$

So U is the optimal solution in V_h^3 according to the energy norm. Take $v = P_h u$.

$$\|u-U\|_E \le \|u-P_hu\|_E = \|(u-P_hu)''\|_{L^2} \le c_1h^2\left\|u^{(4)}\right\|_{L^2}$$

L^2-norm error estimate Since $Lu = u^{(4)}$ with $u(0)=u(1)=u'(0)=u'(1)=0$ is self adjoint, we consider the dual problem.

$$\phi^{(4)} = e, \phi(1) = \phi(0) = \phi'(1) = \phi'(0)$$

$$\|e\|_{L^2}^2 = \int_0^1 e^2dx = \int_0^1 e\phi^{(4)}dx = \int_0^1 e''\phi''dx + [e\phi''' - e'\phi'']_0^1 \implies \|e\|_{L^2}^2 = \int_0^1 e''\phi''dx$$

Using Galerkin Orthogonality, we see that since $P_h\phi \in V_h^3$,

$$\int_0^1 e''(P_h\phi)''dx = 0 \implies \|e\|_{L^2}^2 = \int_0^1 e''(\phi - P_h\phi)''dx$$

$$\|e\|_{L^2}^2 \le \|e''\|_{L^2}\|(\phi-P_h\phi)''\|_{L^2} \le ch^2\left\|u^{(4)}\right\|_{L^2}ch^2\left\|\phi^{(4)}\right\|_{L^2} = ch^4\left\|u^{(4)}\right\|_{L^2}ch^2\|e\|_{L^2}$$

so

$$\|e\|_{L^2} \le Ch^4\left\|u^{(4)}\right\|_{L^2}$$

In practice, we see

$$|\xi_j - u(x_j)| = \mathcal{O}(h^4), \qquad |\eta_j - u'(x_j)| = \mathcal{O}(h^4)$$

7 Abstract Formulation of Conforming Methods for Elliptic Equations

Consider $-(au')' = f, x \in (0,1)$, $u(0) = u(1) = 0$. The variational problem takes $v \in H_0^1$ and seeks a solution to

$$B(u,v) = L(v), B(u,v) = \int_0^1 au'v'dx \text{ (bilinear form) }, L(v) = \int_0^1 \text{ (linear functional)}$$

which was equivalent to the energy minimization problem of finding $u \in H_0^1$ such that $F(u) \le F(v)$ for all $v \in H_0^1$ where the energy functional is defined as

$$F(v) = \frac{1}{2}B(v,v) - L(v) \text{ (energy functional)}$$

So in general, consider the domain $\Omega \subset \mathbb{R}^d$.

7.1 Lax-Milgram Theorem

missing notes Sept 25

So, given a bounded/continuous functional $L(\cdot)$ and a symmetric, bounded, coercive (elliptic?) bilinear form $B(u, v)$,

$$|L(v)| \leq \tilde{L}\|v\| \qquad B(u, v) = B(v, u) \qquad B(u, v) \leq C\|u\| \; \|v\| \qquad B(v, v) \geq c\|v\|^2$$

The variational problem has a unique solution $u \in V$ with the property $\|u\| \; \leq \frac{\tilde{L}}{c}$.

7.2 Abstract Conforming Method

Replace the space V by a finite dimensional subspace $V_h \subset V$. The general Finite Element Method is to find a $u_h \in V_h$ such that $B(u_h, v) = L(v)$ for all $v \in V_h$.
The Lax Milgram theorem tells us there exists a unique solution $u_h \in V_h$ such that $\|u_h\|_V \leq \frac{\tilde{L}}{c}$.
Let $\{phi_i\}_{i=1}^M$ be a basis for V_h where $\dim(V_h) = M$. Let $u_h = \sum\limits_{i=1}^{M} \xi_i \phi_i$, and then let $v = \phi_j$ for $j = 1, ..., M$, so

$$\sum_{i=1}^{M} B(\phi_i, \phi_j)\xi_i = L(\phi_j)$$

for $j = 1, ..., M$. Converting this to a linear algebra problem,

$$A\vec{\xi} = \vec{\mathbf{f}}, a_{ij} = B(\phi_i, \phi_j)$$

We see that A is symmetric and positive definite.

- Note that this is independent of geometry and dimension!

7.2.1 Error

7.2.2 Cea's Lemma

Let V be a Hilbert space and let V_h be a finite dimensional subspace. Given a bilinear form B and a linear functional L satisfying the assumptions of Lax-Milgram, then let u be the solution to the variational probem and let u_h be the solution to the FEM problem, then

$$\|u - u_h\|_V \leq \frac{C}{c}\|u - v\|_V \; \forall \; v \in V_h$$

Thus, u_h is the best possible approximation to u for all functions in V_h with respect to $\|\cdot\|_V$.
Proof. Subtracing the FEM problem from the variational problem,

$$B(u, v) - B(u_h, v) = 0 \; \forall \; v \in V_h \implies B(u - u_h, v) = 0 \; \forall \; v \in V_h$$

Thus we have Galerkin orthogonality with respect to the inner product $B(\cdot, \cdot)$.

$$B(u - u_h, u - u_h) = B(u - u_h, u - v + v - u_h) = B(u - u_h, u - v) \; \forall \; v \in V_h$$

Since $B(u - u_h, v - u_h) = 0$ by Galerkin orthogonality.
Using the assumed properties of this bilinear form,

$$B(u - u_h, u - v) \leq C\|u - u_h\|_V \|u - v\|_V$$

and

$$c\|u - u_h\|_V^2 \leq C\|u - u_h\|_V \|u - v\|_V$$

so

$$\|u - u_h\|_V \leq \frac{C}{c}\|u - v\|_V$$

Note: Error in $\|\cdot\|_V$ Let $v = P_h u$. Therefore

$$\|u - u_h\|_V \leq \frac{C}{C}\|u - P_h u\|_V$$

where $\|u - P_h u\|_V$ is the interpolation error measured in $\|\cdot\|_V$. So convergence occurs if $\|u - P_h u\|_V \to 0$ as $h \to 0$, $M \to \infty$.

Example Given the problem

$$-u'' = f, \qquad u(0) = u(1) = 0$$

Let $I = (0,1)$, then $V = H_0^1(I)$, $B(u,v) = \langle u', v', = \rangle \int_0^1 u'v'dx$, $L(v) = \langle f, v\rangle$. $B(\cdot,\cdot)$ is clearly symmetric.

$$|B(u,v)| \leq \|u'\|_{L^2(I)}\|v'\|_{L^2(I)} \leq \|u'\|_{H_0^1(I)}\|v'\|_{H_0^1(I)}$$

So B is bounded with $C = 1$. Now using Poincare

$$B(v,v) = \|v'\|_{L^2}^2 \geq \frac{1}{2}\left(\|v\|_{L^2(I)} + \|v'\|_{L^2(I)}\right) = \frac{1}{2}\|v\|_V^2$$

So B is coercive ($V-$ ellpitic) with $c = \frac{1}{2}$. f is continuous since

$$|L(v)| = |\langle f, v\rangle| \leq \|f\|_{L^2(I)}\|v\|_{L^2(I)} norm[L^2(I)]f\|v\|_V$$

Using previous results, a cG(1) method has

$$\|u - u_h\|_V \leq 2\|u - P_h u\| \ \leq c_1 h\|u''\|_{L^2(I)}$$

Whereas a cG(2) method has

$$\|u - u_h\|_V \leq c_2 h^2\|u'''\|_{L^2(I)}$$

2D Example Consider the Poisson equation

$$-\Delta u = -(u_{,x,x} + u_{,y,y}) = f, x \in \Omega \subset \mathbb{R}^2 \qquad u = 0, x \in \partial\Omega$$

Assume $f \in L^2(\Omega)$. The variational problem is

$$-\int_\Omega v\Delta u dx = \int_\Omega fv dx$$

Using Green's identities, this becomes

$$\int_\Omega \nabla v \cdot \nabla u dx - \int_{\partial\Omega} v\nabla u \cdot \hat{n} ds = \int_\Omega fv dx$$

If we take $v = 0$ on $\partial\Omega$, then this is

$$\int_\Omega \nabla v \cdot \nabla u dx = \int_\Omega fv dx$$

Thus we have our space $V = H_0^1(\Omega)$. So our variational problem is to find $u \in V$ st $B(u,v) = L(v)$ for all $v \in V$ where

$$B(u,v) = \int\int_\Omega \nabla u \cdot \nabla v dx dy, \qquad L(v) = \int\int_\Omega fv dx dy$$

and the energy is defined as $F(w) = \frac{1}{2}B(w,w) - L(w)$. We note $B(\cdot,\cdot)$ is continuous with $C = 1$ since

$$|B(u,v)| \leq \|\nabla u\|_{L^2}\|\nabla v\|_{L^2} \leq \|\nabla u\|_{H_0^1}\|\nabla v\|_{H_0^1}$$

and it is coercive ($V-$ elliptic) since

$$B(v,v) = \|\nabla v\|_{L^2}^2 \geq \frac{1}{2}\left(C^{-2}\,|v|_{H_0^1}^2 + \|\nabla v\|_{L^2}^2\right) =$$

7.3 Non-Conforming Methods

The two introduction probems are known as **conforming methods**. This is any method where we replace the space V by a finite dimensional subspace $V_h \subset V$. Non-conforming methods have functions outside the space V: $V_h \not\subset V$.

Oct 2 Notes

$$u^{(4)} = f, u(0) = u(1) = u'(0) = u'(1) = 0, I = (0,1)$$

where $f \in L^2(I)$. So our test function space is $V = \mathcal{H}_0^2(I)$ and our problem is

$$B(u,v) = \langle u'', f'' \rangle = \langle f, v \rangle = L(v)$$

B is bounded with $C = 1$, symmetric, and coercivity with $c = \frac{1}{3}$ is proven via Poincare.
Cea's lemma says for cG(3), we have the error estimate

$$\|u - u_h\|_{H^2(I)} \leq ch^2 |u|_{H^4(I)}$$

Biharmonic Equation

$$\Delta^2 u = u_{xxxx} + 2u_{xxyy} + u_{yyyy} = f, x \in \Omega, u|_{\partial\Omega} = 0, \nabla u \cdot \hat{n}|_{\partial\Omega} = 0$$

Multiply by v and integrate over Ω.

$$\int_\Omega v\Delta^2 u dx = \int_\Omega fv dx$$

Using Green's identity,

$$\int_\Omega v\Delta^2 u dx - \int_\Omega \Delta v \Delta u dx = \int_{\partial\Omega} v(\nabla\Delta u \cdot \hat{n}) - \Delta u(\nabla v \cdot \hat{n}) ds$$

Or if $v|_{\partial\Omega} = 0$, $\nabla v \cdot \hat{n}|_{\partial\Omega} = 0$,

$$\int_\Omega v\Delta^2 u dx = B(u,v) = \int_\Omega \Delta v \Delta u dx = \int_\Omega fv dx = L(v)$$

So our space will be $V = \mathcal{H}_0^2$, with energy

$$E[u] = \frac{1}{2}B(u,v) - L(v)$$

Our bilinear form is bounded, symmetric, and coercive (proved with $\|v\|_{H^2} \leq \left\|L^2\right\|_{\Delta v}$). Our functional for f is bounded with $\tilde{L} = \|f\|_{L^2}$.
For 2D, we need 5th order polyomials, thus a cG(5) method. So our approximate V (assuming Ω has a polygon boundary) is

$$V_h = \left\{ v \in H^2(\Omega : v|_{k_j} = \sum_{m,l=0}^{m+l=5} a_j^{lm} x^l y^m, v|_{\tilde{\partial\Omega}} = 0, \nabla v \cdot \hat{n}|_{\tilde{\partial\Omega}} = 0 \right\}$$

We have 6 conditions

- v is continuous on 3 edges
- $\nabla v \cdot \hat{n}$ is continuous on 3 edges

Error estimate from Cea's lemma

$$\|u - u_h\|_{H^2(\Omega)} \leq ch^2 |u|_{H^4(\Omega)}$$

8 2D Mesh Generation

Finite Element Methods have no inherent restriction on the mesh. The simplest and most flexible way to handle the mesh is a simplex. This means a triangulation in 2D and tetrahedrals in 3D. A general simplex is unstructured. Let $K_h = \{K_j\}_{j=1}^M$ That is, the nodes don't necessary lie along the grid points. Let $X_h = \{(x_i, y_i)\}_{i=1}^N$ be the set of **nodes** (vertices of the triangles) and let $S_h = \{s_l\}_{l=1}^E$ be the set of **edges of the triangles. Let h_i be the longest edge of K_j and let $h = \max_{1\leq j \leq M} h_j$, and let ρ_j be the diameter of the largest circle inside K_j. A triangularization is regular if there exists a constant $\alpha > 0$ independent of h such that $\frac{h_j}{\rho_j} << \leq \alpha$ for all j.**

8.1 Mesh Generation Part 1

Given points in $\mathbb{R}^2$, we must form a triangularization. For each point, we can assign an area of influence known as the Voronoi Region V_i. Let X be a set of N points/nodes. For each $\vec{\mathbf{x}}_i \in X$, the Voronoi region is the set of points in $\mathbb{R}^2$ that are at least as close to $\vec{\mathbf{x}}_i$ as any other point in X.

$$V_i = \left\{ \vec{\mathbf{x}} \in \mathbb{R}^2 : \|\vec{\mathbf{x}} - \vec{\mathbf{x}}_i\| \ \leq \|\vec{\mathbf{x}} - \vec{\mathbf{x}}_i\| \ \ \forall \, x_j \in X \right\}$$

That is, it is a collection of half planes defined by the intersection of the half planes $H_{ij} = \left\{ \vec{\mathbf{x}} \in \mathbb{R}^2 : \|\vec{\mathbf{x}} - \vec{\mathbf{x}}_i\| \ \leq \|\vec{\mathbf{x}} - \vec{\mathbf{x}}\right.$ This forms a convex polygonal region V_i which is possibly unbounded.

- **Every point in $x \in \mathbb{R}^2$ has at least one nearest point in X**
- **Two Voronoi regions lie on opposite sides of perpendicular bisector and they never share any interior points. Thus points that belong to two or more regions must lie on a boundary.**
- **Voronoi regions will cover the entire plane.**
- **Voronoi regions along with their edges and vertices form the Voronoi diagram of X.**
- **For a given X, the Voronoi diagram is unique.**

8.1.1 Delaunay Partition

Let X be a set of N points in $\mathbb{R}^2$. Let V be the Voronoi diagram of X. The Delaunay partition is obtained by creating an edge between 2 points in X if and only if their Voronoi regions share a boundary. Mostly, this creates triangles that share a common Voronoi vertex. In the degenerate, 4 or more points share a common Voronoi vertex if the points are co-circular. Degeneracies can always be converted into a triangularization by slicing. The Delauney Triangulation T_h is the Delaunay Partition of a set of N points in $\mathbb{R}^2$ with some strategy to convert degenerate elements into triangles.

- **The Voronoi diagram and the Delaunay triangulation form a duality.**
- **For a given X, V is unique and T_h is unique up to degeneracies.**
- **T_h contains $\mathcal{O}(N)$ triangles for N points.**
- **T_h maximizes the minimum angle.**
- **A circle that circumscribes any triangle in T_h does not contain any $\vec{\mathbf{x}} \in X$ in the interior of the circle.**
- **In C, you can use software from www.qhull.org**

In MATLAB, you can create this via the command t=delaunay(p) which can be plotted via trimesh(t,p(:,1),p(:,2)) where p is your list of points. Your solution u can be plotted via trisurf(t,p(:,1),p(:,2),u). If N is the number of nodes and M is the number of triangle, then t is an $M \times 3$ array of positive integers where for each $t(i,j)$ i is the triangle index (1,...,M) and j is the node index (1,2,3). So the three vertices of triangle k are $p(t(k,1),1:2), p(t(k,2),1:2), p(t(k,3),1:2)$.

8.2 Mesh Generation Part 2

How do we define a domain? One idea: defined a function $\phi(x)$ and consider the level sets of ϕ. Let the boundary be defined by the zero level set. As an example, consider

$$\phi(x) = \|x\|_2 - R \qquad \Omega = \left\{x \in \mathbb{R}^2 : \phi(x) < 0\right\} \qquad \partial\Omega = \left\{x \in \mathbb{R}^2 : \phi(x) = 0\right\}$$

That is, the domain is a disc of radius R centered at the origin with the circle of radius R being the boundary. The exterior of our domain are points such that $\phi(x) > 0$. These level set functions are non unique. For example, $\phi(x) = \|x\|^p - R^p$ for $p \in \mathbb{N}$ describe the same domain and boundary as the above level set function. You can use different norms (even mixed norms) to make various shapes. To make a square, use

$$\phi(x) = \|x\|_\infty - R \qquad \Omega = \left\{x \in \mathbb{R}^2 : \phi(x) < 0\right\} \qquad \partial\Omega = \left\{x \in \mathbb{R}^2 : \phi(x) = 0\right\}$$

Boolean Operators **Consider $\phi_1(x) = \|x\| - R_1$, $\phi_2(x) = \|x\| - R_2$**

$$\phi_3(x) = \min\left\{\phi_1(x), \phi_2(x)\right\}$$

This function produces a negative for any point in either set, so it is easy to see that if ϕ_1 produces Ω_1 and ϕ_2 produces Ω_2, then $\Omega_3 = \Omega_1 \cup \Omega_2$ and $\partial\Omega_3 = \{x : \phi_3(x) = 0\}$
Similarly,

$$\phi_3(x) = \max\left\{\phi_1(x), \phi_2(x)\right\}$$

Produces the intersection of the domains. That is, $\Omega_3 = \Omega_1 \cap \Omega_2$.

Signed Distance Function (SDF) **Note: $\frac{\nabla\phi(x)}{\|\nabla\phi\|}$ is a unit vector that is perpendicular to the level curves of $\phi(x)$.**

$$\frac{\nabla\phi(x)}{\|\nabla\phi\|}\Big|_{\phi=0} = \hat{n}, \text{ outward pointing unit normal on } \partial\Omega$$

So we let ϕ be a level set function. Noting that $\phi|_{\partial\Omega} = 0$, if

$$\phi(x) = \begin{cases} -\operatorname{dist}(x, \partial\Omega) & x \in \Omega \\ \operatorname{dist}(x, \partial\Omega) & x \notin \Omega \end{cases}$$

then $\phi(x)$ is called a signed distance function (SDF).
Example: Consider the annular region defined by $1 < r < 2$. This can be defined by $\phi_1(x) = \max\{1 - r, r - 2\}$ or $\phi_2 = (r-1)(r-2)$, but only $\phi(x)$ is a signed distance function.
If ϕ is an SDF then $\|\nabla\phi\| = 1$ and therefore $\nabla\phi|_{\phi=0}$ is unit normal to $\partial\Omega$. This is because if we have the unit normal and unit tangent vectors to the curve defined by $\phi = 0$, and $\phi = \eta$, where η is the outward pointing normal. Then

$$\nabla\phi = \left(\hat{n}\frac{\partial}{\partial\eta} + \hat{t}\frac{\partial}{\partial\tau}\right)\phi = \hat{n} \implies \|\nabla\phi\| = 1$$

Example. Consider the circular disc of radius a. $\phi_1(x) = r - a$, $\phi_2(x) = r^3 - a^3$ are both valid level set functions for this, but only $\phi_1(x)$ is the SDF since

$$\nabla\phi_1 = \frac{x^2 + y^2}{r^2} = 1 \qquad \phi_2(x) = 9(r-a)^4 \neq 1$$

Let be an SDF. If we want to take any point in $\mathbb{R}^2$ and project it onto the point $y \in \partial\Omega$ that is closest to $\vec{x}$, we can use $\vec{y}$ (point on $\partial\Omega = \vec{x} - \phi(\vec{x})\nabla\phi(\vec{x})$ (starting point- shift amount *unit direction) for $k = 0, 1, 2, ...$.
If a point is equidistant from two points on the boundary, then we have an undefined gradient. In this case, we can project $\vec{x}$ to $y \in \partial\Omega$ via a fixed point iteration

$$\vec{\mathbf{y}}_{k+1} = \vec{\mathbf{y}}_k - \psi(\vec{\mathbf{y}})\frac{\nabla\psi(\vec{\mathbf{y}}_k)}{\|\nabla\psi(\vec{\mathbf{y}}_k)\|}, \vec{\mathbf{y}}_0 = \vec{\mathbf{x}}$$

8.3 Mesh Generation Part 3

Given a level set function ϕ that describes Ω that may or may not be an SDF, how do we generate a triangulation of Ω with some points guaranteed to be on $\partial\Omega$.

8.4 FEM for 2D Poisson

Consider the PDE:

$$-\nabla\cdot(\nabla u) = f, x \in \Omega \qquad u = 0, x \in \partial\Omega$$

The variational problem is: Find $u \in V$ such that

$$\int\int_\Omega \nabla u \cdot \nabla v d\vec{\mathbf{x}} = \int\int_\Omega f v d\vec{\mathbf{x}}$$

for all $v \in V$, where $V = \mathcal{H}_0^1$.
Our approximate V_h is $V_h = \left\{v \in V : v|_{k_j} = a_j + b_j x + c_j y\right\}$ since our elements are triangles. We can let

$$v(x_1, y_1) = v_1 \qquad v(x_2, y_2) = v_2 \qquad v(x_3, y_3) = v_3$$

And then set up the Vandermonde matrix as such:

$$\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

The basis functions ϕ_i have nodes at (x_i, y_i) for $i = 1, ..., N$ and we want them to have the following properties

- $\phi_i(x_k, y_k) = \delta_{ik}$
- $\phi_i|_{k_j}$ **should be linear in x and y**
- $\operatorname{supp}\phi_i$ **is all triangular elements for which (x_i, y_i) is a vertex**
- ϕ_i **is continuous on Ω (automatically satisfied by previous properties)**

So we can write our basis

$$v(x, y) = \sum_{i=1}^{N} v(x_i, y_i)\phi_i(x, y)$$

So our finite element problem is to find $U \in V_h^1$ s.t.

$$\int\int_\Omega \nabla U \cdot \nabla v d\vec{\mathbf{x}} = \int\int_\Omega f v d\vec{\mathbf{x}} \qquad \forall\, v \in V_h^1$$

Let $U = \sum_{j=1}^N \xi_j \phi_j(x,y)$ and $v = \phi_i$ for $i = 1,..,N$

$$\sum_{i=1}^M \xi_i \int\int_\Omega \nabla\phi_i \cdot \nabla\phi_j d\vec{\mathbf{x}} = \int\int_\Omega f\phi_j d\vec{\mathbf{x}} = \sum_k \int\int_k f\phi_j d\vec{\mathbf{x}}$$

where we handle the right hand side via $\mathcal{O}(h^2)$ numerical integration

$$\sum_k \int\int_k f\phi_j d\vec{\mathbf{x}} = \frac{1}{3}A_k(f(\vec{\mathbf{x}}_1)\phi_i(\vec{\mathbf{x}}_1) + f(\vec{\mathbf{x}}_2)\phi_i(\vec{\mathbf{x}}_2) + f(\vec{\mathbf{x}}_3)\phi(\vec{\mathbf{x}}_3))$$

where A_k is the area of element K and $\vec{\mathbf{x}}_i$ are the vertices of element K.

8.4.1 Poisson Eqn on the Unit square $[0,1]\times[0,1]$

For a uniform Triangulation, we fill the unit square with squares of side length h and draw diagonals from northwest to southeast to obtain the triangular elements. We can index the interior nodes $1,...,m^2$ where the grid spacing is $h = \Delta x = \Delta y = \frac{1}{m+1}$. The number of unknowns is equal to the number of interior nodes which is $N = m^2$. The basis vanishes on the boundary, so we don't include those as unknowns.

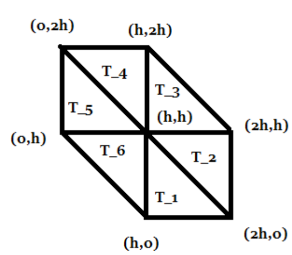
$$\sum_{j=1}^{m^2} \xi_j \langle \nabla\phi_i, \nabla\phi_j\rangle = \langle f, \phi_i\rangle$$

for $i = 1,...,m^2$. So let's build the stiffness matrix.

- **If $i = j$, $a_{ii} = \langle\nabla\phi_i, \nabla\phi_i\rangle = \int\int_{L_i} \|\nabla\phi_i\|^2 d\vec{\mathbf{x}}$ where L_i is the support of ϕ_i. We end up integrating over 6 triangular elements. On each one, if $\phi|_{T_k} = a_k + b_k x + c_k y$ then $\nabla\phi|_{T_k} = \begin{pmatrix} b_k \\ c_k \end{pmatrix}$ so $norm\nabla\phi_i{}^2 = (b_k^2 + c_k^2)$ and if the area of each triangle is $A_k = \frac{1}{2}h^2$,**

  $$\int\int_{T_k} \|\nabla\phi_i\|^2 d\vec{\mathbf{x}} = \frac{1}{2}h^2\left(b_k^2 + c_k^2\right)$$

  For each triangle we just need to find b_k and c_k. We can do this for a generic setup.

  After the details, we get that $a_{ii} = 4$.

- **If $j = i - m$ then the overlap is $T_1 and T_6$.**

$$\int\int_{T_1} \langle \nabla\phi_i, \nabla\phi_{i-m}\rangle \, d\vec{\mathbf{x}} = \frac{1}{2}h^2(a\alpha + b\beta) = -\frac{1}{2}$$

$$\int\int_{T_6} \langle \nabla\phi_i, \nabla\phi_{i-m}\rangle \, d\vec{\mathbf{x}} = -\frac{1}{2}$$

  So $a_{i,i-m} = -1$

- $j = i + m$**. By symmetry,** $a_{i,i+m} = -1$
- $j = i - m + 1$**: overlap is** T_1 **and** T_2**. For each of these, we end up with integration of orthogonal elements, so** $a_{i,i-m+1} = 0$
- $j = i + m + 1$**: by symmetry** $a_{i,i+m+1} = 0$
- $a_{i,i+1} = -1$
- $a_{i,i-1} = -1$

Therefore we have the linear system $A\xi = b$ **where** A **is** $N \times N$ **(recall** $N = m^2$**) and symmetric, positive definite. For the case** $N = 9$**,**

$$A = \begin{pmatrix} D & -I & \\ -I & D & -I \\ & -I & D \end{pmatrix} \qquad D = \begin{pmatrix} 4 & -1 & \\ -1 & 4 & -1 \\ & -1 & 4 \end{pmatrix}$$

So we see this is SPD and block diagonal. The Cholesky factorization followed by forward and backward substitution provides a fast way to do this. Another option is the conjugate gradient method (pcg in MATLAB)
For the RHS, $b_i = \langle f, \phi_i \rangle$**. This involves an integral over six triangles.**

$$b_i = \int\int_{T_1} + ... + \int\int_{T_6} f\phi_i dx$$

We employ a quadrature

$$b_i = \sum_{k=1}^{6} \frac{h^2}{6}\left(f(x_i)\phi_i(x_i) + f(x_j)\phi_i(x_j) + f(x_l)\phi_i(x_l)\right)$$

Which when we use $\phi_i(x_j) = \delta_{ij}$**, the simplifies to**

$$b_i = \sum_{k=1}^{6} \frac{h^2}{6} f(x_i) = h^2 f(x_i)$$

This is $\mathcal{O}(h^2)$**, same as FDM.**

Arbitrary Geometry   What if we have an arbitrary geometry? We need to produce a list of nodes $p(1:N, 1:2)$ **and a list of triangles** $t(1:M, 1:3)$**. Assume that** P **has been sorted so that if** NIN **is the number of interior nodes,** $p(1:NIN, 1:2)$ **are the interior nodes** $p(NIN+1:N, 1:2)$ **are the boundary nodes. This is true for the course web code. Check 'assemblestiff' and 'assembleRHS' for these procedures.**

Nonhomogenous Dirichlet BCs

$$-\nabla \cdot (\nabla u) = f, x \in \Omega \qquad u = g(x), x \in \partial\Omega$$

We consider $V_g = \left\{ v \in \mathcal{H}^1, v = g \; \forall \; x \in \Omega \right\}$ The variational problem becomes: Find $u \in V_g$ st $\langle \nabla u, \nabla v \rangle = \langle f, v \rangle$ for all $v \in V_0$. The FEM becomes to let

$$U = \sum_{N_j \in \partial\Omega} g(N_j)\phi_j(x) + \sum_{N_j \in \Omega \setminus \partial\Omega}$$

where $N_j = (x_j, y_j)$ are the coordinates for the jth node. Therefore we need to solve

$$\sum_{N_j \in \Omega \setminus \partial\Omega} \xi \langle \nabla \phi_i, \nabla \phi_j \rangle = \langle f, \phi_i \rangle - \sum_{N_j \in \partial\Omega} g(N_j)\phi_j(x)$$

where $\{i : N_i \in \Omega \setminus \partial\Omega\}$

Robin and Neumann BCs

$$-\nabla \cdot (\nabla u) = f, x \in \Omega \qquad u = g(x), x \in \partial\Omega_1, \qquad \nabla u \cdot \hat{n} + \kappa u = h(x), x \in \partial\Omega_2,$$

where $\partial\Omega = \partial\Omega_1 + \partial\Omega_2$, $\hat{n}$ is the outward pointing normal, and $\kappa \geq 0$.
The variational problem is

$$\langle f, v \rangle = -\langle \nabla \cdot \nabla u, v \rangle$$

Using Green's identities,

$$\langle f, v \rangle = \langle \nabla u, \nabla v \rangle - \oint_{\partial\Omega} \nabla u \cdot \hat{n} v ds$$

or

$$\langle f, v \rangle = \langle \nabla u, \nabla v \rangle - \oint_{\partial\Omega_1} \nabla u \cdot \hat{n} v ds - \oint_{\partial\Omega_2} \nabla u \cdot \hat{n} v ds$$

We let our test functions be zero on $\partial\Omega_1$, that is $v(x) = 0, x \in \partial\Omega_1$, and let $V_g = \left\{ v \in H^1(\Omega) : v = g, x \in \partial\Omega_1 \right\}$. So our variational problem is to find $u \in V_g$ st

$$\langle f, v \rangle = \langle \nabla u, \nabla v \rangle - \oint_{\partial\Omega_2} (h - \kappa u) \cdot \hat{n} v ds$$

or rather

$$\langle f, v \rangle + \oint_{\partial\Omega_2} h v ds = \langle \nabla u, \nabla v \rangle - \oint_{\partial\Omega_2} \kappa u v ds$$

for all $v \in V_0$.

Error Analysis **Homogenous Dirichlet BCs. Recall Cea's lemma.**

$$\|u - u_h\|_{\mathcal{H}^1(\Omega)} \leq C \|u - v\|_{\mathcal{H}^1} \; \forall \; V \in V_h$$

Let $v = P_h u$ and use $\|u - P_h u\|_{H^1} \leq Ch \, |u|_{H^2}$ (essentially, the energy norm estimate), where

$$|v|^2_{H^2} = \int \int_{\Omega} (v_{xx})^2 + 2(v_{xy})^2 + (v_{yy})^2 d\vec{\mathbf{x}}$$

Regarding the poisson equation, if either Ω is a convex polygon or if $\partial\Omega$ is a smooth curve, then the solution to the weak form of the Poisson equation on Ω satisfies

$$\|u\|_{H^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}$$

Error Estimate Let H be a Hilbert space with the norm $\|\|_H$ and a scalar product $\langle \cdot, \cdot \rangle$. Let the sobolev embedding V into H (more regular into less regular) be continuous:

$$\|v\|_H \leq \|v\|_V$$

Let $B(u, v)$ be a symmetric, bounded, V-ellpitic bilinear form with

$$B(v, v) \geq \beta_* \|v\|_V^2 \qquad |B(u, v)| \leq \beta^* \|u\|_V \|v\|_V$$

Let $L(\cdot)$ be a linear functional that is bounded with $|L(v)| \leq \tilde{L}\|v\|_V$. Let $u \in V$ and $u_n \in V_h$ be solutions to $B(u, v) = L(v)$ and $B(u_n, v) = L(v)$ respectively. Then

$$\|u - u_h\|_H \leq C\|u - u_h\|_V \sup_{\psi \in \mathcal{H} \setminus \{0\}} \frac{\inf_{v \in V_h} \|\phi - v\|_V}{\|\psi\|_H}$$

where for a given $\psi \in H$, $\phi \in V$ is the solution to $B(\phi, w) = \langle \psi, w \rangle$ for all $w \in V$ (the dual problem). The $\|\cdot\|_H$ of a function $f \in H$ can be written as

$$\|f\|_H = \sup_{\psi \in H \setminus \{0\}} \frac{\langle \psi, f \rangle}{\|\psi\|_H}$$

Because a function $u \in V$ is also in H, and because $u - u_h \in V$, we get

$$\|u - u_h\|_H = \sup_{\psi \in H \setminus \{0\}} \frac{\langle \psi, u - u_h \rangle}{\|\psi\|_H}$$

where ($-\Delta\phi = \psi$) But

$$\langle u - u_h, \psi \rangle = B(u - u_n, \phi) = B(u - u_n, \phi) - B(u - u_h, v)$$

(the last term is zero by Galerkin orthogonality) where $v \in V_h$. so then

$$|\langle u - u_h, \psi \rangle| = |B(u - u_n, \phi - v)| \leq C\|u - u_n\|_V \|\phi - v\|_V = C\|u - u_n\|_V \inf_{v \in V_h} \|\phi - v\|_V$$

by boundedness of $B(\cdot, \cdot)$ Therefore we use this in place of $\langle \psi, f \rangle$ in the above to get the desired result.

L^2 Error Estimate for cG(1) on 2D poisson Eqn w/ Homogenous Dirichlet BC Let $u \in H_0^1$ satisfy $\langle \nabla u, \nabla v \rangle = \langle f, v \rangle$ for all $v \in H_0^1$. Let $V_h^1 = \left\{ v \in H_0^1 : v|_K = P_1(k), k \in T_h \right\}$, where T_h is a triangulation of Ω that for every $h > 0$ satisfy mesh regularity condition:

$$\frac{\text{(longest edge of K)}}{\text{(diameter of largest inscribed circle of K)}} \leq \alpha \; \forall \; K \in T_h$$

Let $u_h \in V_h^1$ satisfy $\langle \nabla u_h, \nabla v \rangle = \langle f, v \rangle$ for all $v \in V_h^1$ then $\|u - u_h\|_{L^2} \leq ch^2 \, |u|_{H^2(\Omega)}$.
Proof: Using the previous lemma with $V = H^1$ and $H = L^2$,

$$\|u - u_h\|_{L^2} \leq c_1 \|u - u_h\|_{H^1} \sup_{\psi \in L^2 \setminus \{0\}} \frac{\inf_{v \in V_h^1} \|\phi - v\|_{H^1}}{\|\psi\|_{L^2}}$$

ψ is RHS of the dual problem $B(\phi, w) = \langle \psi, w \rangle$ but by the previous theoprem $\|\phi\|_{H^2} \leq C_2 \|\psi\|_{L^2}$, $\|\psi\|_{L^2} \geq c_2^{-1} \|\phi\|_{H^2}$ and $\inf_{v \in V_h^1} \|\phi - v\| \leq \|\phi - P_h \phi\|_{H^2} \leq c_3 h \|\phi\|_{H^2}$.

$$\|u - u_h\|_{L^2} \leq c_1 c_2 c_3 h \|u - P_h u\|_{H^1} \sup_{\psi \in L^2 \setminus \{0\}} \frac{\|\phi\|_{H^2}}{\|\phi\|_{H^2}}$$

so

$$\|u - u_h\|_{L^2} \leq ch \|u - u_h\|_{H^1} \leq ch^2 \, |u|_{H^2}$$

9 1D Heat Equation

$$\begin{cases} u_t - u_{xx} = 0 & 0 < x < 1, 0 < t < T \\ u(t,0) = u(t,1) = 0 \\ u(0,x) = u_0(x), 0 < x < 1 \end{cases}$$

Exact solution is

$$u(x,t) = \sum_{k=1}^{\infty} a_k e^{-k^2\pi^2 t} \sin(k\pi x) \qquad a_k = \sqrt{\frac{2}{\pi}} \int_0^1 u_0(x) \sin(k\pi y) dy$$

If $u_0 \in L^2(0,1)$ then the series converges uniformly for $t > 0$.

Stability Estimates **All nodes decay:**

$$\|u(t,\cdot)\|_{L^2(0,1)} \leq \|u_0\|_{L^2(0,1)} \qquad \forall\, 0 < t < T$$

All nodes of the t-derivative decay

$$\|u_t(t,\cdot)\|_{L^2(0,1)} \leq \frac{c}{t} \|u_0\|_{L^2(0,1)} \qquad \forall\, 0 < t < T$$

9.1 Semi-Discrete FEM

Let $\Omega \subset \mathbb{R}^2$.

$$u_t - \Delta u = f \qquad (0,T] \times \Omega \qquad u = 0, x \in (0,T] \times \partial\Omega \qquad u(0,\vec{\mathbf{x}}) = u_0(\vec{\mathbf{x}}), x \in \Omega$$

Let $V = \mathcal{H}_0^1(\Omega)$. Seek $u \in (0,T] \to V$ such that

$$\langle u_t, v \rangle + \langle \nabla u, \nabla v \rangle = \langle f, v \rangle \qquad \forall\, v \in V, t \in (0,T]$$

with

$$\langle u(0), v \rangle = \langle u_0, v \rangle \qquad \forall\, v \in V, t \in (0,T]$$

So then our Semi-Discrete FEM lets us replace V by $V_h \subset V$. Let $\phi_1(\vec{\mathbf{x}}), ..., \phi_N(\vec{\mathbf{x}})$ be a basis for V_h. Then

$$u(t,\vec{\mathbf{x}}) = \sum_{k=1}^{N} u_j(t) \phi_N(\vec{\mathbf{c}})$$

Which gives rise to a linear system of ODES once we put this into the variational problem

$$\sum_{k=1}^{N} \langle \phi_i, \phi_j \rangle \dot{u}_j + \sum_{k=1}^{N} \langle \nabla\phi_i, \nabla\phi_j \rangle u_j = \langle f, \phi_i \rangle \qquad i = 1, ..., N$$

along with the initial conditions

$$\sum_{k=1}^{N} \langle \phi_i, \phi_j \rangle u_j(0) = \langle u_0, \phi_i \rangle \qquad i = 1, ..., N$$

This can be expressed as a matrix equation with $G_{ij} = \langle \phi_i, \phi_j \rangle$ (known as the mass matrix, which is SPD) along with the usual stiffness matrix $A_{ij} = \langle \nabla\phi_i, \nabla\phi_j \rangle$ and the forcing vector $f_i = \langle f, \phi_i \rangle$. So we have the linear constant coefficient system of ODES

$$\begin{cases} G\dot{\vec{\mathbf{u}}} + A\vec{\mathbf{u}} = \vec{\mathbf{f}} \\ G\vec{\mathbf{u}}(0) = \vec{\mathbf{u}}_0 \end{cases}$$

We can solve this using the Cholesky Factorization $G = R^T R$ where R is upper triangular. Let $\vec{\mathbf{w}} = R\vec{\mathbf{u}}$ so that $\vec{\mathbf{u}} = R^{-1}\vec{\mathbf{w}}$, so we now have the system

$$\begin{cases} \dot{\vec{\mathbf{w}}} + (R^{-1})^T A R^{-1}\vec{\mathbf{w}} = (R^{-1})^T\vec{\mathbf{f}} \\ \vec{\mathbf{w}}(0) = (R^{-1})^T\vec{\mathbf{u}}_0 \end{cases}$$

Now let $\hat{A} = (R^{-1})^T A R^{-1}$ and $\vec{\mathbf{g}} = (R^{-1})^T\vec{\mathbf{f}}$. $\hat{A}$ is symmetric because $\hat{A}^T = ((R^{-1})^T A R^{-1})^T = (R^{-1})^T A R^{-1} = \hat{A}$ and is positive definite because for $\vec{\mathbf{w}} \neq 0$

$$\vec{\mathbf{w}}^T\hat{A}\vec{\mathbf{w}} = \vec{\mathbf{w}}^T(R^{-1})^T A(R^{-1})\vec{\mathbf{w}} = \vec{\mathbf{u}}^T A\vec{\mathbf{u}} > 0$$

$$\begin{cases} \dot{\vec{\mathbf{w}}} + \hat{A}\vec{\mathbf{w}} = \vec{\mathbf{g}} \\ \vec{\mathbf{w}}(0) = (R^{-1})^T\vec{\mathbf{u}}_0 \end{cases}$$

So we can solve this exactly in time. However computationally, this is ludicrously expensive and impractical since we already have error from the FEM anyway.

$$\vec{\mathbf{w}}(t) = e^{-\hat{A}t}\vec{\mathbf{w}}(0) + \int_0^t e^{-\hat{A}(t-s)}\vec{\mathbf{g}}(s)ds$$

Error Analysis for ? (missing notes 11/4)

$$\|u(t) - u_h(t)\|_{L^2} \leq \|u_0 - P_h u_0\|_{L^2} + ch^2\left(\|?\|\ \right)$$

Proof: Let $e(t) = u(t) - u_h(t)$ and write

$$e(t) = (u(t) - R_h u(t)) + (R_h u(t) - u_h(t)) = \rho(t) + \theta(t)$$

where $\rho(t) \in V$ and $\theta(t) \in V_h^1$. From the elliptic error analysis: $\|\rho(t)\|_{L^2} \leq ch^2\|u(t)\|_{H^2}$ but $u(t) = u(0) + \int_0^t u_t(s)ds$. Therefore

$$\|\rho(t)\|_{L^2} \leq ch^2\left(\|u_0\|_{L^2} + \int_0^t \|u_t(s)\|_{H^2}ds\right)$$

We also note that

$$\langle u_t, v\rangle + \langle \nabla u, \nabla v\rangle = \langle f, v\rangle \ \forall\ v \in V_h^1 \subset V$$

using the Ritz projection: $\langle \nabla u, \nabla v\rangle = \langle \nabla R_h u, \nabla v\rangle$. Therefore

$$\langle u_t - R_h u_t + R_h u_t, v\rangle + \langle \nabla R_h u, \nabla v\rangle = \langle f, v\rangle \ \forall\ v \in V_h^1$$

So from this we subtract our FEM $\langle u_{h,t}, v\rangle + \langle \nabla u_h, \nabla v\rangle = \langle f, v\rangle$ for all $v \in V_h^1$ so recalling $\rho = u - R_h u, \theta = R_h u - u_h$

$$\langle \rho_t, v\rangle + \langle \theta_t, v\rangle + \langle \nabla\theta, \nabla v\rangle = 0$$

for all $v \in V_h^1$. but $\theta \in V_h^1$, so we can take $v = \theta$:

$$\langle \theta_t, \theta \rangle + \|\nabla \theta\|_{L^2}^2 = -\langle \rho_t, \theta \rangle$$

but

$$\langle \theta_t, \theta \rangle = \frac{1}{2}\frac{d}{dt}\|\theta(t)\|_{L^2}^2 = \|\theta(t)\|_{L^2}\frac{d}{dt}\|\theta(t)\|_{L^2}$$

also,

$$-\langle \rho_t, \theta \rangle \leq \|\rho_t\|_{L^2}\|\theta\|_{L^2}$$

Therefore

$$\frac{d}{dt}\|\theta(t)\|_{L^2} \leq \|\rho_t\|_{L^2}$$

If we integrate from 0 to t

$$\|\theta(t)\|_{L^2} - \|\theta(0)\|_{L^2} \leq \int_0^t \|\rho_t(s)\|_{L^2} ds$$

or

$$\|\theta(t)\|_{L^2} \leq \|\theta(0)\|_{L^2} + \int_0^t \|\rho_t(s)\|_{L^2} ds$$

But

$$\|\theta(0)\|_{L^2} = \|R_h u_0 - u_0 + u_0 - P_h u_0\|_{L^2} \leq \|R_h u_0 - u_0\|_{L^2} + \|u_0 - P_h u_0\|_{L^2} \leq ch^2\|u_0\|_{H^2} + \|u_0 - P_h u_0\|_{L^2}$$

And

$$\|\rho_t\|_{L^2} = \|u_t - R_h u_t\|_{L^2} \leq ch^2\|u_t\|_{H^2}$$

Recall that the ritz projection has the property $\langle \nabla R_h u, \nabla v \rangle = \prod \nabla u \nabla v$ and the projection onto the mesh has the property $\langle u_h, v \rangle = \langle u, v \rangle$.

$$\|\theta(t)\|_{L^2} \leq \|u_0 - P_h u_0\|_{L^2} + ch^2\left(\|u_0\|_{H^2} + \int_0^t \|u_t(s)\|_{H^2} ds\right)$$

$$\|\rho(t)\|_{L^2} \leq ch^2\left(\|u_0\|_{H^2} + \int_0^t \|u_t(s)\|_{H^2} ds\right)$$

$$\|u(t) - u_h(t)\|_{L^2} \leq \|\rho(t)\|_{L^2}\|\theta(t)\|_{L^2}$$

Backward Euler **Instead of solving exactly in time, we can use time stepping methods. Consider the variational problem**

$$\left\langle \frac{u_h^n - u_h^{n-1}}{dt}, v \right\rangle + \langle \nabla u_h^n, \nabla v \rangle = \langle f^n, v \rangle$$

for all $v \in V^h$ and for all $n = 1, ..., K$. where $\Delta t = \frac{T}{K}$.

$$\langle u_h^0, v \rangle = \langle u_0, v \rangle \ \forall\, v \in V^h$$

$$u_h^n = \sum_{j=1}^N u_j^n \phi_j \qquad v = \phi_i \ \forall\, i = 1, ..., N$$

In matrix form, we have our mass matrix G and stiffness matrix A

$$G = \langle \phi_i, \phi_j \rangle \qquad A = \langle \nabla \phi_i, \nabla \phi_j \rangle \qquad f = \langle f, \phi_i \rangle \qquad u_0 = \langle u_0, \phi_i \rangle$$

Therefore

$$Gu^n - Gu^{n-1} + \Delta t A u^n = dt f^n \qquad Gu^0 = u_0$$

or

$$(G + dtA)\, u^n = Gu^{n-1} + dt f^n \qquad Gu^0 = u_0$$

Stability Assume $f=0$. Take $v=u_h^n$ in variational problem

$$\langle u_h^n, u_h^n\rangle - \left\langle u_h^{n-1}, u_h^n\right\rangle + \Delta t \left\langle \nabla u_h^n, \nabla u_h^n\right\rangle = 0$$

$$\|u_h^n\|_{L^2}^2 - \left\langle u_h^{n-1}, u_h^n\right\rangle + \Delta t\|\nabla u_h^n\|_{L^2}^2 = 0$$

but $\left\langle u_h^{n-1}, u_h^n\right\rangle \leq \left\|u_h^{n-1}\right\|_{L^2}\|u_h^n\|_{L^2} \leq \frac{1}{2}\|u_h^n\|_{L^2}^2 + \frac{1}{2}\left\|u_h^{n-1}\right\|_{L^2}$ **so using this in the above formula,**

$$\|u_h^n\|_{L^2} \leq \left\|u_h^{n-1}\right\|_{L^2} \leq ... \leq \left\|u_0^0\right\|_{L^2} \leq \|u_0\|_{L^2}$$

So the Backward Euler Continuous Galerkin Method is (BEcG) is unconditionally L^2 stable for any $\Delta t \geq 0$.

Theorem Condtions? notes 11/6/2014

$$\|u_h^n - u(t^n)\|_{L^2} \leq ch^2\left(\|u_0\|_{H^2} + \int_0^{t^n}\|u_t(s)\|_{H^2}ds\right) + \Delta t\int_0^{t_n}\|u_{tt}(s)\|_{L^2}ds$$

Note this means $\|u_h^n - u(t^n)\|_{L^2} = \mathcal{O}(h^2+\Delta t)$, second order in space, first order in time.
Proof: Let $\rho(t) = u(t) - R_h u(t)$. Then

$$u(t_n) - u_n^h = e(t^n) = u(t^n) - R_h u(t^n) + R_h u(t^n) - u_n^h = \rho(t^n) + \theta(t^n)$$

Note: θ is only defined at $t^n = n\Delta t$ for $n = 1, ..., K$, while ρ is defined for all $t \in [0,T]$ while ρ is defined for all $t \in [0,T]$. As in the semi-discrete case:

$$\|\rho(t^n)\|_{L^2} \leq ch^2\left(\|u_0\|_{H^2} + \int_0^{t^n}\|u_t(s)\|_{H^2}ds\right)$$

And the Ritz projection is $\langle\nabla u, \nabla v\rangle = \langle\nabla R_h u, \nabla v\rangle$ for all $v \in V_h^1$. Therefore

$$\langle u_t, v\rangle + \langle\nabla R_h u, \nabla v\rangle = \langle f, v\rangle$$

or in the discrete case

$$\left\langle R_h\frac{u(t^n) - u(t^{n-1})}{\Delta t}, v\right\rangle + \langle\nabla R_h u, \nabla v\rangle = \langle f, v\rangle + \langle w^n, v\rangle \qquad (STAR)$$

where $w_n = R_h\frac{u(t^n)-u(t^{n-1})}{\Delta t} - u_t(t^n)$. Note that this is true for all $v \in V_h^1$ Now we introduce the FEM:

$$\left\langle\frac{u_h^n - u_h^{n-1}}{\Delta t}, v\right\rangle + \langle\nabla u_h^n, \nabla v\rangle = \langle f, v\rangle \ \forall\ v \in V_h^1$$

Subtract this from star, and define $\theta(t^n) = R_h u(t^n) - u_h^n$

$$\left\langle\frac{\theta^n - \theta^{n-1}}{\Delta t}, v\right\rangle + \langle\nabla\theta^n, \nabla v\rangle = \langle w^n, v\rangle$$

Let $v = \theta^n$. So we have

$$\left\langle\frac{\theta^n - \theta^{n-1}}{\Delta t}, \theta^n\right\rangle + \langle\nabla\theta^n, \nabla\theta^n\rangle = \langle w^n, \theta^n\rangle$$

or

$$\|\theta^n\|_{L^2}^2 - \left\langle\theta^n, \theta^{n-1}\right\rangle \leq \Delta t\|w^n\|_{L^2}\|\theta^n\|_{L^2}$$

but $\langle \theta^n, \theta^{n-1} \rangle \leq \|\theta^n\|_{L^2} \|\theta^{n-1}\|_{L^2}$

$$\|\theta^n\|_{L^2} - \|\theta^{n-1}\|_{L^2} \leq \Delta t \|w^n\|_{L^2}$$

Summing this inequality from $n = 1, ..., l$**,**

$$\|\theta^l\|_{L^2} - \|\theta^0\|_{L^2} \leq \Delta t \sum_{n=1}^{l} \|w^n\|_{L^2}$$

now we write

$$w^n = w_1^n + w_2^n = (R_h - I)\left(\frac{u(t^n) - u(t^n - \Delta t)}{\Delta t}\right) + \left(\frac{u(t^n) - u(t^n - \Delta t)}{\Delta t} - u_t(t^n)\right)$$

So then estimating each part,

$$w_1^n = (R_h - I)\frac{1}{\Delta t}\int_{t^n - dt}^{t^n} u_t(s)ds$$

or

$$\|w_1^n\|_{L^2} = \frac{1}{\Delta t}\int_{t^n - dt}^{t^n} \|(R_h - I)u_t(s)\|_{L^2} ds \leq \frac{ch^2}{\Delta t}\int_{t^n - dt}^{t^n} \|u_t(s)\|_{H^2} ds$$

so

$$\Delta t \sum_{n=1}^{l} \|w_1^n\|_{L^2} \leq ch^2 \int_0^{t^l} \|u_t(s)\|_{H^2} ds$$

Also

$$\Delta t w_2^n = (u(t^n) - u(t^n - \Delta t) - \Delta t u_t(t^n)) = \int_{t^n - dt}^{t^n} (t^{n-1} - s) u_{tt}(s) ds$$

Where $z(t) = (t^{n-1} - t) u_{tt}(t)$ **and**

$$\int_{t^{n-1}}^{t^n} z(t)dt = \int_{t^n - \Delta t}^{t_n} (t^{n-1} - s) u_{tt}(s) ds = -\Delta t u_t(t^n) + \int_{t^{n-1}}^{t^n} u_t(s) ds = -\Delta t u_t(t^n) + u(t^n) - u(t^n - \Delta t)$$

So since $(t^{n-1} - s) \leq \Delta t$

$$\Delta t \sum_{n=1}^{l} \|w_2^n\| \ \leq \Delta t \int_{t^n - dt}^{t^n} \|u_{tt}(s)\|_{L^2} ds$$

Lastly, we see

$$\|\theta^0\|_{L^2} = \|R_h u_0 - u_h^0\|_{L^2} = \|R_h u_0 - P_h u^0\|_{L^2} \leq \|R_h u_0 - u_0\|_{L^2} + \|u_0 - P_h u^0\|_{L^2} \leq ch^2 \|u_0\|_{H^2}$$

Therefore

$$\|\theta^n\|_{L^2} \leq ch^2 \|u_0\|_{H^2} + c_2 h^2 \int_0^{t^n} \|u_t(s)\|_{H^2} ds + \Delta t \int_0^{t_n} \|u_{tt}(s)\|_{L^2} ds$$

Finally,

$$\|u_h^n - u(t^n)\|_{L^2} \leq \|\theta^n\|_{L^2} + \|\rho^n\|_{L^2} \leq c_3 h^2 \left(\|u^0\|_{H^2} + \int_0^{t_n} \|u_t(s)\|_{H^2} ds\right) + \Delta t \int_0^{t^n} \|u_{tt}(s)\|_{L^2} ds$$

9.2 Crank-Nicolson Method

9.3 Stiff Problems

9.4 Nonconfroming FEMs

Consider

$$-\nabla \cdot \nabla u = f, x \in \Omega, \qquad u = 0, x \in \partial\Omega$$

Find $u \in H_0^1(\Omega)$ such that

$$\langle \nabla u, \nabla v \rangle = \langle f, v \rangle \ \forall \ v \in \mathcal{H}_0^1(\Omega)$$

Conforming FEMs replace V by a finite dimensional subspace V_h such as $V_h = \{v \in \mathcal{H}_0^1 v|_k = a_k + b_k x + c_k y\}$. Nonconforming FEMs consider finite dimensional spaces that are not subspaces of V. For example, the Crouzeix-Raviart element space (1973) is

$V_h = \{v \in L^2(\Omega) : v|_k = a_k + b_k x + c_k y,$ **v is continuous at the midpoints of interior edges, and v=0 at the midpoint of**

Note: Each element has 3 degrees of freedom and 3 constraints (otherwise the problem might be ill-posed).
Clearly, $V_h \notin V$ and has less regularity than V. Therefore we need to reinterpret the Bilinear form as

$$B(u,v) = \int\int_\Omega \nabla u \cdot \nabla v dx \text{ turns into } B_h(u,v) = \sum_{k \in T_h} \int\int_K \nabla u \cdot \nabla v dx$$

Note this $B_h(\cdot,\cdot)$ is symmetric, and

$$B_h(u,u) = \sum_{k \in T_h} \|\nabla u\|^2_{L^2(K)} > 0, \ \forall \ u \in V_h \setminus \{0\}$$

So our Variational problem is: Find u_h such that

$$\sum_{k \in T_h} \langle \nabla u_h, \nabla v \rangle_K = \langle f, v \rangle \ \forall \ v \in V_h$$

What is our new basis? Let $\vec{e}_j$ denote that midpoint of edge j, where $j = 1, ..., S$, where S is the number of edges. Let the edges be ordered such that $j = 1, ..., \hat{S}$ refer to the interior edges, and let $j = \hat{S}, ..., S$. Since $u = 0$ on the boundary,

$$\phi_i(e_j) = \delta_{ij} \implies u_h = \sum_{j=1}^{\hat{S}} u_j \phi_j$$

Use test functions $v = \phi_i$ for $i = 1, ..., \hat{s}$, so the matrix form is

$$A\vec{\mathbf{u}} = \vec{\mathbf{f}}, \vec{\mathbf{u}} = (u_1, ..., u_{\hat{S}})^T$$

where $f_i = \langle f, \phi_i \rangle$, $A_{ij} = B_h(\phi_i, \phi_j)$. Symmetry of B_h implies symmetry of A. We also see

$$\vec{\mathbf{v}}^T A \vec{\mathbf{v}} = \sum_{i=1}^{\hat{S}} v_i (A\vec{\mathbf{v}})_i = \sum_{i=1}^{\hat{S}} \sum_{j=1}^{\hat{S}} v_i A_{ij} v_j = \sum_{i=1}^{\hat{S}} \sum_{j=1}^{\hat{S}} v_i B_h(\phi_i, \phi_j) v_j$$

By linearity

$$\vec{\mathbf{v}}^T A \vec{\mathbf{v}} = B_h(\sum_{i=1}^{\hat{S}} v_i \phi_i, \sum_{j=1}^{\hat{S}} v_j \phi_j = B_h(v_h, v_h) = \sum_{k \in T_h} \|\nabla v_h\|^2_{L^2(K)} > 0 \ \forall \ v_h \in V_h \setminus \{0\}$$

Therefore A is SPD.

9.4.1 L^2 error estimate

$$\|u-u_h\|_{L^2(\Omega)} + h\sum_{k\in T_h}\|\nabla(u-u_h)\|^2_{L^2(K)} \leq ch^2\|u\|_{H^2}$$

9.4.2 2D Biharmonic Eqn

$$\Delta^2 u = f, x\in\Omega \qquad u = \nabla u\cdot\hat{n} = 0, x\in\partial$$

Let $\mathcal{H}_0^2(\Omega) = \left\{v\in H^2(\Omega) : v = \nabla u\cdot\hat{n} = 0, x\in\partial\Omega\right\}$. The variational problem is to find $u\in H_0^2(\Omega)$ such that

$$\langle\nabla u,\nabla v\rangle = \langle f, v\rangle\ \forall\ v\in\mathcal{H}_0^2(\Omega)$$

In 2D on triangles, the minimum degree polynomial in $\mathcal{H}_0^2(\Omega)$ is 5. So we consider the modified bilinear form (the one for the plate problem). Let

$$\tilde{B}(u,v) = B(u,v) + (1-\sigma)S(u,v) \qquad \text{, where } B(u,v) = \langle\Delta u,\Delta v\rangle_{L^(\Omega)}$$

where $0<\sigma\leq\frac{1}{2}$ is Poisson's ratio. And $S(u,v) = -\langle\Delta u,\Delta v\rangle + \langle u_{xx}, v_{xx}\rangle + 2\langle u_{xy}, v_{xy}\rangle + \langle u_{yy}, v_{yy}\rangle$. Using appropriate integration by parts,

$$S(u,v) = \oint_{\partial\Omega}\left(\frac{\partial^2 u}{\partial\hat{n}\partial\hat{t}}\cdot\frac{\partial v}{\partial\hat{t}} - \frac{\partial^2 u}{\partial\hat{t}^2}\frac{\partial v}{\partial\hat{n}}\right)ds$$

where $\frac{\partial v}{\partial\hat{n}} = \nabla v\cdot\hat{n}$ and $\frac{\partial v}{\partial\hat{t}} = \nabla v\cdot\hat{t}$. $\hat{n}$ is unit normal to $\partial\Omega$. $\hat{t}$ is unit tangent to $\partial\Omega$.

$$\frac{\partial^2 u}{\partial\hat{n}\partial\hat{t}} = \sum_{i=1}^{2}\sum_{j=1}^{2}???$$

$$\frac{\partial^2 u}{\partial\hat{t}^2} = \sum_{i=1}^{2}\sum_{j=1}^{2}???$$

For $v\in H_0^2(\Omega)$, $v=0$ on $\partial\Omega$ so $\nabla v\cdot\hat{t}=0$ on $\partial\Omega$. $\nabla v\cdot\hat{n}=0$ on $\partial\Omega$. Therefore $S(u,v)=0$. Using IBP, $u\in H^3(\Omega), v\in H^2(\Omega)$, something?
Therefore the modified variational problem is to find $u\in H_0^2(\Omega)$ such that

$$\tilde{B}(u,v) = \langle f, v\rangle$$

Note that $\tilde{B}(\cdot,\cdot)$ is symmetric. It is also continuous since

$$\left|\tilde{B}(u,v)\right| \leq \sigma\|\Delta u\|_{L^2}\|\Delta v\|_{L^2} + (1-\sigma)\|u\|_{H^2}\|v\|_{H^2} \leq \sigma\|u\|_{L^2}\|v\|_{L^2} + (1-\sigma)\|u\|_{H^2}\|v\|_{H^2} \leq \|u\|_{H^2}\|v\|_{H^2}$$

This bilinear form is also $V-$elliptic

$$\tilde{B}(u,v) = \sigma\|\Delta v\|^2_{L^2} + (1-\sigma)\,|v|^2_{H^2} \geq \beta_*\|v\|^2_{H^2}$$

Therefore the modified variational problem satisfies the Lax-Milgram Theorem. In fact, if $u\in\mathcal{H}^3(\Omega)$, then solutions with $B(\cdot,\cdot)$ and $\tilde{B}(\cdot,\cdot)$ will be identical.

The Equivalent minimization problem is to find $u\in H_0^2$ st

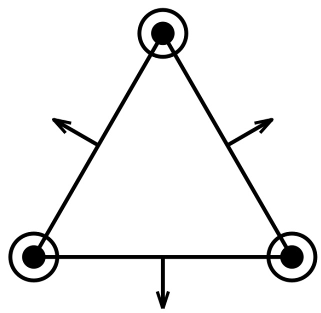
$$\tilde{F}(u)\leq\tilde{F}(v)\ \forall\ v\in H_0^2(\Omega)$$

where

$$\tilde{F}(v) = \frac{1}{2}\tilde{B}(v,v) - \langle f, v\rangle$$

The nonconforming FEM   with the (Morley Element) is:
Let V_h be the set $v \in L^2(\Omega)$ such that

- $v_k = P_2(k)$**, that is** $v|_k = a_{00} + a_{10}x + a_{01}y + a_{11}xy + a_{20}x^2 + a_{02}y^2$
- v **is continuous on interior nodes**
- $v = 0$ **on bdy nodes**
- $\nabla v \cdot \hat{n}$ **is continuous on the midpoints of interior edges**
- $\nabla v \cdot \hat{n} = 0$ **on the midpoints of the bdy edges**



Let

$$\tilde{B}_h(u,v) = \sum_{k \in T_h} \sigma \left\langle \Delta u, \Delta v \right\rangle_K + (1-\sigma)\left(\left\langle u_{xx}, v_{xx} \right\rangle_K + 2\left\langle u_{xy}, v_{xy} \right\rangle_K + \left\langle u_{yy}, v_{yy} \right\rangle_K \right)$$

The Variational Problem   is to find $u_h \in V_h$ such that

$$\tilde{B}_h(u,v) = \langle f, v \rangle \ \forall \, v \in V_h$$

This is solvable because if we let the discrete norm on this space be

$$\|v_h\|_h^2 = \sum \left(\left\langle v_{xx}, v_{xx} \right\rangle_K + 2\left\langle v_{xy}, v_{xy} \right\rangle_K + \left\langle v_{yy}, v_{yy} \right\rangle_K \right)$$

then

$$\tilde{B}_h(u,u) = \sum_{k \in T_h} \sigma \left\langle \Delta u_h, \Delta u_h \right\rangle + (1-\sigma)\|u_h\|_h^2 > 1-\sigma)\|u_h\|_h^2 \ \forall \, u \in V_h$$

So $\tilde{B}_h(\cdot,\cdot)$ is symmetric and positive definite as long as $0 < \sigma < 1$.

The L^2-error analysis   prove by Lascauz and Le Saint in 1978.

$$\|u - u_h\|_{L^2(\Omega)} \le ch^2 \left(|u|_{H^3(\Omega)} + h|u|_{H^4(\Omega)} \right)$$

Note: The Morley element methid is $\mathcal{O}(h^2)$ accurate and has far fewer degrees of freedom than the $\mathcal{O}(h^4)$ $\mathcal{H}_0^2(\Omega)$ conforming FEM.

Lemma: Strang's Second Lemma

- **Let V be a Hilbert space such that $V = \mathcal{H}_0^r(\Omega)$ for some integer $r > 0$. Let $B(\cdot,\cdot) : V \times V \to \mathbb{R}$ be a bilinear form that satisfies assumptions of Lax-Milgram. Let $L : V \to \mathbb{R}$ be a linear functional bounded on $\|\cdot\|_V$.**
- **Let V_h be a finite dimensional space with norm $\|\cdot\|_h$. Let $\tilde{B}_h(\cdot,\cdot) : V_h \times V_h \to \mathbb{R}$ be a discrete bilinear form that satisfies**
  - $B_h(u,v) = B_h(v,u)$ **for all** $u, v \in V_h$
  - $|B_h(u,v)| \leq \beta^* \|u\|_h \|v\|_h$ **for all** $u, v \in V_h \cup V$ **(both spaces!)**
  - $B_h(u,u) \geq \beta_* \|u\|_h^2$ **for all** $u \in V_h$ **(analogous to** $V-$**ellipticity)**

where $\beta^*, \beta_* > 0$ **and are indepenedent of** h**. Let** $L_h : V_h \to \mathbb{R}$ **be a linear functional bounded in** $\|\cdot\|_h$**. If** $u \in V$ **is the unique solution to** $B(u,v) = \langle f, v\rangle$ **for all** $v \in V$ **and** $u_h \in V_h$ **is the unique solution of** $B_h(u_h, v) = L_h(v)$ **for all** $v \in V_h$**, then there exists a constant** $c > 0$ **independent of** h **such that**

$$\|u - u_h\|_h \leq c \left(\inf_{v \in V_h} \|u - u_h\|_h + \sup_{w \in V_h, w \neq 0} \frac{|B_h(u,w) - L_h(w)|}{\|w\|_h} \right)$$

This is a generalization of Cea's lemma. The extra (second) piece comes from the fact that we are using a modified bilinear form that isn't exactly correct. This second piece vanishes in the conforming methods, and is known as the consistency error. The first term is the approximation error.
Proof of this error estimate: For any $v \in V_h$**:**

$$\begin{aligned}
\beta_* \|u_h - v\|_h^2 &\leq B_h(u_h - v, u_h - v) = B_h(u_h - u + u - v, u_h - v) \\
&\leq B_h(u - v, u_h - v) + B_h(u_h, u_h - v) - B_h(u, u_h - v) \\
&\leq B_h(u - v, u_h - v) + L_h(u_h - v) - B_h(u, u_h - v) \\
\beta_* \|u_h - v\|_h &\leq \frac{B_h(u - v, u_h - v)}{\|u_h - v\|_h} + \frac{L_h(w) - B_h(u,w)}{\|w\|_h} \\
&\leq \beta^* \|u - v\|_h + \sup_{w \in V_h, w \neq 0} \frac{L_h(w) - B_h(u,w)}{\|w\|_h} \\
\beta_* \left(\|u_h - v\|_h + \|u - v\|_h \right) &\leq (\beta^* + \beta_*) \|u - v\|_h + \sup_{w \in V_h, w \neq 0} \frac{L_h(w) - B_h(u,w)}{\|w\|_h}
\end{aligned}$$

Using the triangle inequality, on the left hand side, we have

$$\begin{aligned}
\beta_* \|u_h - u\|_h &\leq (\beta^* + \beta_*) \|u - v\|_h + \sup_{w \in V_h, w \neq 0} \frac{L_h(w) - B_h(u,w)}{\|w\|_h} \\
\|u_h - u\|_h &\leq c \left(\|u - v\|_h + \sup_{w \in V_h, w \neq 0} \frac{L_h(w) - B_h(u,w)}{\|w\|_h} \right) \\
\|u_h - u\|_h &\leq c \left(\inf_{v \in V_h} \|u - v\|_h + \sup_{w \in V_h, w \neq 0} \frac{L_h(w) - B_h(u,w)}{\|w\|_h} \right)
\end{aligned}$$

Where taking the infimum comes from the fact that the second to last line is true for all $v \in V_h$**.**

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10 Mixed FEM

For the Poisson problem,

$$\int_0^1 wu'dx = \int_0^1 fwdx$$

Variational Problem : Find $u \in V$ and $p \in W$ such that $u = -p'$ and

$$\int_0^1 uvdx = \int_0^1 pv'dx \qquad \int_0^1 wu'dx = \int_0^1 fwdx \;\forall\; v \in V, w \in W$$

This seems backwards, u is more regular than p.
Claim: The Variational Problem is equivalent to a saddle-point problem (as opposed to a minimization). We want to find $u \in V$ and $p \in W$ such that

$$F(u,w) \leq F(u,p) \leq F(v,p) \;\forall\; v \in V, w \in W$$

where

$$F(v,w) = \frac{1}{2}\int_0^1 v^2dx - \int_0^1 v'wdx + \int_0^1 fwdx$$

That is we maximize with respect to the second argument and minimize with respect to the first argument.

Proof of Problem equivalence **Suppose $u \in V$ and $p \in W$ are solutions to the variational problem**

$$\langle u,v\rangle - \langle v',p\rangle = 0 \;\forall\; v \in V \qquad \langle u',w\rangle = \langle f,w\rangle \;\;\forall\; w \in W$$

Let $\tau = v - u \in V$. If $F(v,w) = \frac{1}{2}\langle v,v\rangle - \langle v',w\rangle + \langle f,w\rangle$,

$$\begin{aligned}
F(v,p) =& F(u+\tau,p) = \frac{1}{2}\langle u+\tau,u+\tau\rangle - \langle u'+\tau',w\rangle + \langle f,w\rangle \\
=& \frac{1}{2}\langle u,u\rangle + \langle u,\tau\rangle + \frac{1}{2}\langle \tau,\tau\rangle \\
& - \langle u',w\rangle - \langle \tau',w\rangle + \langle f,w\rangle \\
=& \left(\frac{1}{2}\langle u,u\rangle - \langle u',p\rangle + \langle f,p\rangle\right) + (\langle u,\tau\rangle - \langle \tau',p\rangle) + \frac{1}{2}\langle \tau,\tau\rangle \\
=& F(u,p) + \frac{1}{2}\langle \tau,\tau\rangle
\end{aligned}$$

Since the second set of parentheses disappear via the variational problem. Therefore

$$F(v,p) \geq F(u,p) \;\forall\; v \in V$$

For the second part of this proof, let $\tau = w - p \in W$,

$$F(u,w) = F(u,p+\tau) = F(u,p) - \langle u',\tau\rangle + \langle f,\tau\rangle = F(u,p)$$

therefore

$$F(u,p+\tau) = F(u,p)$$

So then we only need to minimize in the first argument. The second argument will not change the energy at this minimum for the Poisson problem.

$$F(u,w) = F(u,p) \leq F(v,p) \;\forall\; v \in V, w \in W$$

Now we need to prove the converse equivalence. Suppose $u \in V$ and $p \in W$ satisfy

$$F(u,w) \leq F(u,p) \leq F(v,p) \; \forall \; v \in V, w \in W$$

So for a minimizer u, let G be defined as

$$F(u,p) \leq F(u+\epsilon v, p) = G(\epsilon) \; \forall \; v \in V, \epsilon \in \mathbb{R}$$

So then

$$G(\epsilon) = F(u+\epsilon v, p) = \frac{1}{2}\langle u,u\rangle + \epsilon \langle u,v\rangle + \frac{1}{2}\epsilon^2 \langle v,v\rangle - \langle u',p\rangle - \epsilon \langle v',p\rangle + \langle f,p\rangle$$

$G(0)$ is a minimum, so then we expect that $G'(0)$ is a critical point.

$$G'(\epsilon) = \langle u,v\rangle + \epsilon \langle v,v\rangle - \langle v',p\rangle$$

So then at $\epsilon = 0$,

$$\langle u,v\rangle = \langle v',p\rangle$$

Secondly,

$$G(\epsilon) = F(u, p+\epsilon w) \leq F(u,p)$$

$$G(\epsilon) = \frac{1}{2}\langle u,u\rangle - \langle u',p\rangle - \epsilon \langle u',w\rangle + \langle f,p\rangle + \epsilon \langle f,w\rangle$$

$$G'(\epsilon) = -\langle u',w\rangle + \langle f,w\rangle$$

So at $\epsilon = 0$,

$$\langle u',w\rangle = \langle f,w\rangle$$

Thus we have proven equivalence

10.1 Finite Element Method

Let T_h be the mesh with $\{x_i\} = \{x_0, x_1, ..., x_M, x_{M+1}\}$. Let $I_i = (x_{i-1}, x_i)$ with $h_i = x_i - x_{i-1}$. We let V_h be piecewise linear and let W_h be piecewise constant

$$V_h = \left\{ v \in H^1(0,1) : v|_{I_i} = a_i + b_i x \right\}$$

$$W_h = \left\{ w \in L^2(0,1) : w|_{I_i} \right\} = c_i$$

Note that we do not impose the boundary conditions in the space. Boundary conditions are weakly enforced via the variational problem.

The FEM is to find $u_h \in V_h$ and $p_h \in W_h$ such that

$$\langle u_h, v\rangle = \langle v', p_h\rangle \;\; \forall \; v \in V_h \qquad \langle u_h', w\rangle = \langle f, w\rangle \;\; \forall \; w \in W_h$$

Claim: There exists a unique solution to FEM. Proof: It suffices to show that if $f = 0$ then the only solution is $u_h = p_h = 0$. Let $f = 0$ and take $v = u_h$, $w = p_h$. So

$$\langle u_h, u_h\rangle = \langle u_h', p_h\rangle \qquad \langle u_h', p_h\rangle = 0 \implies \|u_h\|_{L^2} = 0 \implies u_h = 0$$

This implies

$$\langle v', p_h\rangle = 0 \; \forall \; v \in V_h$$

So pick v such that $v' = 0$ for all elements except one. This means $p_h = 0$ on that element. Repeat this process for all elements to conclude that $p_h = 0$.
SO we note that the FEM is equivalent to a saddle point problem: find $u_h \in V_h$ and $p_h \in W_h$ such that

$$F(u_h, w) \leq F(u_h, p_h) \leq F(v, p_h) \; \forall \; v \in V_h, w \in W_h$$

Mixed FEM Basis **Let $\phi_i \in V_h$ such that $\phi_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ This is the hat function just like we had in the cG1 method. However, we also have that $\psi_i \in W_h$, $\psi_i(x_j) = \begin{cases} 1 & x \in (x_{i-1}, x_i) \\ 0 & \text{elsewhere} \end{cases}$. Set**

$$u_h = \sum_{j=0}^{M+1} u_j \phi_j \qquad p_h = \sum_{k=1}^{M+1} p_k \psi_k$$

The u_h bases are based on nodal values (values at each x_i)- there are $M+2$ of these. The p_h live inside the elements, based on modal values- there are $M+1$ of these.

Formulation

$$\sum_{j=0}^{M+1} \langle \phi_i, \phi_j \rangle u_i - \sum_{k=1}^{M+1} \langle \phi_i', \psi_k \rangle p_k = 0$$

$$\sum_{j=0}^{M+1} \left\langle \phi_j', \phi_l \right\rangle u_j = \langle f, \psi_l \rangle$$

So let $A \in \mathbb{R}^{(M+2)\times(M+2)}$ and $B \in \mathbb{R}^{(M+2)\times(M+1)}$ where

$$a_{ij} = \langle \phi_i, \phi_j \rangle \qquad i = 0, ..., M+1 \qquad j = 0, ..., M+1$$
$$b_{ij} = \langle \phi_i', \psi_k \rangle \qquad i = 0, ..., M+1 \qquad k = 1, ..., M+1$$

So we have the system

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} \vec{\mathbf{u}} \\ \vec{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \vec{\mathbf{0}} \\ -\vec{\mathbf{f}} \end{pmatrix}$$

This matrix is symmetric but not positive definite. It has positive and negative eigenvalues, but it can be shown that 0 is not an eigenvalue, so we have a unique solution. If we use a uniform mesh, we see that

$$a_{0,0} = a_{M+1,M+1} = \frac{h}{3} \qquad a_{i-1,i} = a_{i+1,i} = \frac{h}{6} \qquad a_{ii} = \frac{4h}{6} \qquad i = 1, ..., M$$
$$b_{k,k} = 1 \qquad a_{k+1,k} = -1 \qquad k = 0, 1, ..., M$$

L^2 error analysis

$$\|p - p_h\|_{L^2} + \|u - u_h\|_{L^2} \leq ch\|p\|_{H^2}$$

and

$$\|u - u_h\|_{L^2} \leq ch^2\|p\|_{H^2}$$

This is the opposite result as what we found for the cG(1) method.

10.1.1 Multi Dimensions

$$-\Delta p = f \text{ in } \Omega \qquad p = 0 \text{ on } \partial\Omega$$

Let $\vec{\mathbf{u}} = -\nabla p$ so $-\nabla \cdot \nabla p = \nabla \cdot u = \vec{\mathbf{f}}$.
Define the spaces $\vec{\mathbf{V}} = \vec{\mathbf{H}}(div, \Omega) = \left\{ \vec{\mathbf{v}} \in \left[L^2(\Omega)\right]^2 : \nabla \cdot \vec{\mathbf{v}} \in L^2(\Omega) \right\}$. $W = L^2(\Omega)$. Let $\vec{\mathbf{v}} \in \vec{\mathbf{V}}$ be a test function

$$\int\int_\Omega \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} d\vec{\mathbf{x}} = -\int\int_\Omega \vec{\mathbf{v}} \cdot \nabla d\vec{\mathbf{x}} = \int\int_\Omega \nabla \cdot \vec{\mathbf{v}} p d\vec{\mathbf{x}} - \oint_{\partial\Omega} p\vec{\mathbf{v}} \cdot \hat{n} ds$$

Since $p = 0$ on the boundary, we have

$$\int\int_{\Omega} \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} d\vec{\mathbf{x}} = \int\int_{\Omega} \nabla \cdot \vec{\mathbf{v}} p d\vec{\mathbf{x}}$$

So the variational problem is to find $\vec{\mathbf{u}} \in \vec{\mathbf{V}}$ and $p \in W$ such that

$$\langle \vec{\mathbf{u}}, \vec{\mathbf{v}} \rangle - \langle \nabla \cdot \vec{\mathbf{v}}, p \rangle = 0 \ \forall \ \vec{\mathbf{v}} \in V$$

$$\langle \nabla \cdot \vec{\mathbf{u}}, w \rangle = \langle f, w \rangle \ \ \forall \ \vec{\mathbf{w}} \in W$$

We now consider the Raviart-Thomas (1977) elements. They are triangular elements where we let w be constant on the triangles

$$W_h = \left\{ w \in L^2(\Omega) : w|_k = a_k \textbf{ (constant) } \right\}$$

V_h is trickier since the divergence needs to exist across the element edges. One idea is to use

$$V_h = \left\{ \vec{\mathbf{v}} \in L^2(\Omega) : v|_k = \begin{pmatrix} a_0 + a_1 x + a_2 y \\ b_0 + b_1 x + b_2 y \end{pmatrix} \right\}$$

Note that $\nabla \cdot \vec{\mathbf{v}}|_k = a_1 + b_2$ which is a constant. This has 2 many degrees of freedom. So we have the additional constraint that the outward pointing normals are constant on each edge: $\vec{\mathbf{v}} \cdot \hat{n}|_e =$ constant. This gives us the constraint

$$\vec{\mathbf{v}}|_k = \begin{pmatrix} a_0 + a_1 x \\ b_0 + a_1 y \end{pmatrix}$$

So our element space is

$$V_h = \left\{ \vec{\mathbf{v}} \in L^2(\Omega) : v|_k = \begin{pmatrix} a_0 + a_1 x \\ b_0 + a_1 y \end{pmatrix} \right\}$$

Proof: $\vec{\mathbf{n}}_1 = (y_3 - y_2, x_2 - x_3)$ $\vec{\mathbf{n}}_1 \perp (x_3 - x_2, y_3 - y_2)$**,** $\vec{\mathbf{n}}_2 = (y_1 - y_3, x_3 - x_1)$ $\vec{\mathbf{n}}_2 \perp (x_1 - x_3, y_1 - y_3)$**,** $\vec{\mathbf{n}}_3 = (y_2 - y_1, x_1 - x_2)$ $\vec{\mathbf{n}}_3 \perp (x_2 - x_1, y_2 - y_1)$**. Note that $\vec{\mathbf{n}}_1 + \vec{\mathbf{n}}_2 + \vec{\mathbf{n}}_3 = \vec{\mathbf{0}}$.**

$$e_1 : x_1(s) = x_2 + s(x_3 - x_2) \qquad y_1(s) = y_2 + s(y_3 - y_2)$$

$$e_2 : x_2(s) = x_3 + s(x_1 - x_3) \qquad y_2(s) = y_3 + s(y_1 - y_3)$$

$$e_3 : x_3(s) = x_1 + s(x_2 - x_1) \qquad y_3(s) = y_1 + s(y_2 - y_1)$$

without loss of generality: $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (0, y_2)$. Constraints

$$\frac{d}{ds} \left[\vec{\mathbf{v}}(x_l(s), y_l(s)) \cdot \hat{n} \right] = 0 \qquad l = 1, 2, 3$$

$$E_1 : x_3(y_3 - y_2)a_1 + (y_3 - y_1)^2 a_2 - b_1 x_3^2 + x_3(y_2 - y_3)b_2 = 0$$

$$E_2 : x_3 y_3 a_1 + y_3^2 a_2 - b_1 x_3^2 + x_3 y_3 b_2 = 0$$

$$E_2 : ???a_1 = 0$$

$a_2 = 0$**. Subtract E_2 from E_1 to get that $b_2 = a_1$. Then E_2 tells us that $b_1 = 0$. Note: The Edge to Element inversion: $\vec{\mathbf{v}} \cdot \hat{n}_l = E_l$ for $l = 1, 2, 3$ can uniquely determine $\vec{\mathbf{v}}|k = \begin{pmatrix} a_0 + a_1 x \\ b_0 + a_1 y \end{pmatrix}$.**

$$a_0 = \frac{-1}{2\,|K|} \sum_{l=1}^{3} x_l E_l \qquad b_0 = \frac{-1}{2\,|K|} \sum_{l=1}^{3} y_l E_l \qquad c_0 = \frac{-1}{2\,|K|} \sum_{l=1}^{3} E_l$$

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Where $|K|$ is the area of the element K. Therefore

$$\nabla \cdot v|_k = 2a_1 = \frac{1}{|K|}\sum_{l=1}^{3} E_l$$

So the divergence is piecewise constant.
So the FEM is to find $\vec{\mathbf{u}}_h \in \vec{\mathbf{V}}_h$ and $p_h \in W_h$ st

$$\langle \vec{\mathbf{u}}_h, \vec{\mathbf{v}} \rangle - \langle \nabla \cdot \vec{\mathbf{v}}, p_h \rangle = 0 \;\forall\; \vec{\mathbf{v}} \in \vec{\mathbf{V}}_h$$

$$\langle \nabla \cdot \vec{\mathbf{u}}_h, w \rangle = \langle f, w \rangle \;\;\forall\; w \in W_h$$

where

$$W_h = \left\{ w \in L^2(\Omega) : w|_k = a_k \textbf{ (constant) } \right\}$$

$$V_h = \left\{ \vec{\mathbf{v}} \in H(div, \Omega) : v|_k = \begin{pmatrix} a_k + b_k x \\ c_k + b_k y \end{pmatrix} \right\}$$

Basis **Let $\vec{\phi}_l \in \vec{\mathbf{V}}_h$ for $l = 1, ..., S$ where S is the total number of edges. Where $(\vec{\phi}_l \cdot \hat{n})(\vec{\mathbf{e}}_k) = \delta_{lk}$ where $\vec{\mathbf{e}}_k$ is the midpoint of the kth edge.**

$$\vec{\mathbf{u}}_h = \sum_{l=1}^{S} u_j \vec{\phi}_j(\vec{\mathbf{x}})$$

With $u_j = (\vec{\mathbf{u}}_h \cdot \hat{n}_l)(\vec{\mathbf{e}}_l)$ (u_h dotted with the normal and evaluated at the midpoint).

$$\vec{\mathbf{u}}_h = \sum_{l=1}^{S} u_j \vec{\phi}_j(\vec{\mathbf{x}})$$

Let $\psi_i \in W_h$ for $i = 1, .., M$ where M is the total number of elements.

$$\vec{\psi}_i(\vec{\mathbf{x}}) = \begin{cases} 1 & \vec{\mathbf{x}} \in k_i \\ 0 & \vec{\mathbf{x}} \notin k_i \end{cases}$$

$$p_n = \sum_{i=1}^{M} p_i \psi_i(\vec{\mathbf{x}})$$

where $p_i = \frac{1}{|k_i|}\int\int_{K_i} p_h d\vec{\mathbf{x}}$ (the average value of the pressure over the element).
no class on dec 4 nov 2 hw 3 due presentations begin dec 11

11 Appendix

Areas **To calculate the area of a triangle given the vertices $\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2, \vec{\mathbf{x}}_3$,**

$$area = \frac{1}{2}\|(\vec{\mathbf{x}}_3 - \vec{\mathbf{x}}_1) \times (\vec{\mathbf{x}}_2 - \vec{\mathbf{x}}_1)\|$$