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Part I
Ordinary Differential Equation Theory

1 Introductory Theory

An n^{th} order ODE for $y = y(t)$ has the form

$$F(t, y', y'', .., y^{(n)}) = 0 \qquad \text{(Implicit Form)}$$

Usually it can be written

$$y^{(n)} = f(t, y', y'', .., y^{(n-1)}) \qquad \text{(Explicit Form)}$$

A solution y is defined on $y : I \to \mathbb{R}$ with $y \in C^n(I)$ for some $I \subseteq \mathbb{R}$ such that

$$y^{(n)}(t) = f(t, y'(t), y''(t), .., y^{(n-1)}(t)) \qquad \forall\ t \in I$$

- n is the **order** of the ODE. It is the highest derivative to appear in the equation.
- The ODE is **linear** if F depends linearly on $y, ..., y^{(n)}$

$$y^{(n)} = g(t) + \sum_{i=0}^{n-1} \alpha_i(t) y^{(i)}(t)$$

  and is said to be **homogenous** if $g(t) \equiv 0$.
- The ODE is **nonlinear** if F depends nonlinearly on $y^{(n)}$
- If the solution is defined on whole of $\mathbb{R}$ then we call it a **global solution**
- If the solution is defined on a subinterval of $\mathbb{R}$ then we call it a **local solution**

1.1 Senses of Solutions

1.1.0.1 Classical Solution
$u' = f$ in a **classical sense** if $u \in C^1$ and $u'(x) = f(x)\ \forall\ x$

1.1.0.2 Weak Solution
$u' = f$ in a **weak sense** if $u \in L^1_{\text{loc}}$ and $u' = f$ in $\mathcal{D}'$ sense.
Classical solutons are always also weak solutions

1.1.0.3 Distributional Solution
$u' = f$ in a **distributional sense** if $u \in \mathcal{D}'$ and $u' = f$ in $\mathcal{D}'$ sense.
Classical solutions and weak solutions are always also distributional solutions

1.1.0.4 Regularity of Solutions For $u' = 0$ all solutions are classical, weak, and distributional solutions.
For $xu' = 0$ the solution $u = \delta$ is neither classical nor weak.
Thus, the regularity of the solution depends on the DE.

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1.1.1 Initial Value Problems (IVP)

A problem is an IVP if it is given in the form

$$\begin{gathered} y^{(n)} = f(t, y', y'', .., y^{(n-1)}) \\ y(a) = \gamma_0 \\ y'(a) = \gamma_1 \\ \vdots \\ y^{(n-1)} = \gamma_{n-1} \end{gathered}$$

where a is the lower boundary of the domain.

- Linear IVPs have a unique solution.
- **Existence of Solution**
  - **Local Existence Theorem or Peano Existence Theorem**: If f is continuous on $\mathbb{R}^n$, then every $(t_0, u_0, ..., u_0^{(n-1)})$ there exists an open interval $(t_0 - \epsilon, t_0 + \epsilon) = I \subset \mathbb{R}$ with $\epsilon > 0$ that contains t_0 and there exists a continuously dfferentiable function $u : I \to \mathbb{R}$ that satisfies the IVP.
  - **Local Existence Theorem** If f is continuous in a neighborhood of $(a, \gamma_0, ..., \gamma_{n-1})$ there exists an open interval $(t_0 - \epsilon, t_0 + \epsilon) = I \subset \mathbb{R}$ with $\epsilon > 0$ that contains t_0 and there exists a continuously dfferentiable function $u : I \to \mathbb{R}$ that satisfies the IVP.
- **Uniqueness of Solution**
  - **Uniqueness by Continuous Differentiability of** f: If ∇f is continuous (if f is continuously differentiable), then the solution is unique.
  - **Uniqueness by Lipschitz**: If $f(u, t)$ is Lipschitz continuous in u then the solution is unique.
- **Gronwall's Inequality**: For $u(t)$ continous and $\phi(t) \geq 0$ continuous defined on $0 \leq t \leq T$ and u_0 is a constant, if $u(t)$ satisfies

$$u(t) \leq u_0 + \int_0^t \phi(s)u(s)ds \qquad \text{for } t \in [0, T]$$

$$\text{then,} \qquad u(t) \leq u_0 exp\left(\int_0^t \phi(s)ds\right) \qquad \text{for } t \in [0, T]$$

A generalization allows $u_0 = \mu(t)$ to depend on time. Then

$$u(t) \leq \mu(t) + \int_{t_0}^t v(s)u(s)ds \implies u(t) \leq \mu(t) + \int_{t_0}^t \mu(s)v(s)e^{\int_s^t v(z)dz}ds$$

Also, if we consider going backward in time, (again u_0 constant)

$$u(t) \leq u_0 + \int_t^{t_0} v(s)u(s)ds, t \leq t_0 \implies u(t) \leq u_0 e^{\int_t^{t_0} v(s)ds}$$

1.1.2 Boundary Value Problems (BVP)

- A BVP with **separated conditions** affect multiple endpoints such as in the form

$$g_a(y(a)) = 0,\ g_b(y(b)) = 0$$

- A BVP with **unseparated conditons** affect the endpoints simultaneously, such as in the periodic conditions

$$y(a) - y(b) = 0$$

1.1.3 Systems of ODEs

One could also consider solutions to systems of ODEs.
Any ODE can be converted into a first order system of ODEs. Example:

$$x''' = t + \cos(x'')e^x \qquad \text{becomes} \qquad y' = \begin{pmatrix} x' \\ x'' \\ x''' \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ t + \cos y_3 e^{y_2} \end{pmatrix}$$

1.2 Linear Equations

Linear equations are linear in y, and have the form

$$\sum_{j=0}^{n} a_j(t) D^{(j)} y(t) = g(t)$$

$$Ly = g(t)$$

Otherwise the ODE is **nonlinear** for some $a_0, a_1, ..., a_n, g$ and $D^k = \dfrac{d^k}{(dt)^k}$. Note that $a_j(t)$ need not be linear.

1.2.1 General solutions

For ($a_1, a_2, ..., a_n$ continuous; g continuous; $a_n \neq 0$), linear ODEs have infinitely many solutions of the form

$$y(t) = y_p(t) + \sum_{j=1}^{n} c_j y_j(t)$$

Where ($y_1, y_2, ..., y_n$) are linearly independent solutions to $Ly = 0$ and $y_p(t)$ is a particular solution to $Ly = g$.

- Linear IVPs have a unique solution.

1.2.1.1 Linear Systems of ODEs Any system of linear ODEs can be viewed as the matrix equation

$$\vec{\mathbf{y}}' = A\vec{\mathbf{y}}$$

with solution

$$\vec{\mathbf{y}} = e^{At}\vec{\mathbf{c}} \qquad \text{where } x(t) = y_1(t)$$

For a vector of arbitrary constants c determined by intial or boundary values.

- If A is diagonalizable , $A = VDV^{-1}$ with its eigenvectors V, then $e^{At}c = Ve^{Dt}V^{-1}\vec{\mathbf{c}}$. Since $V = (\vec{\mathbf{v}}_1 \ \ldots \ \vec{\mathbf{v}}_n)$ and we can define arbitrary constants $\vec{\mathbf{d}} = V^{-1}\vec{\mathbf{c}}$, this becomes

  $$\vec{\mathbf{y}}(t) = e^{At}c = d_1 e^{\lambda_1 t}\vec{\mathbf{v}}_1 + d_2 e^{\lambda_2 t}\vec{\mathbf{v}}_2 + ... + d_n e^{\lambda_n t}\vec{\mathbf{v}}_n$$

  Alternatively you can simply evaluate $Ve^{Dt}V^{-1}\vec{\mathbf{c}}$ and take the first component.

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1.2.2 Integraton Methods

1.2.2.1 Integrating Factor

Given

$$y'(x) + p(x)y(x) = q(x)$$

Multiplying by

$$y'(x)e^{\int pdx} + p(x)e^{\int pdx}y(x) = q(x)e^{\int pdx}$$

Integrating both sides is used with reverse product rule

$$y(x)e^{\int pdx} = \int q(x)e^{\int pdx}dx + c_1$$

1.2.2.2 Variation of Parameters

Given

$$y'' + q(t)y' + r(t)y = g(t)$$

We find the solutions to the associated homogenous equation ($y'' + q(t)y' + r(t)y = 0$)

$$y_c = (t) = c_1y_1(t) + c_2y_2(t)$$

And we want to find a particular solution to $y'' + q(t)y' + r(t)y = g(t)$ in the form

$$y_p = (t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

We let

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0 \qquad \text{Condition 1}$$

and so

$$y_p' = (t) = u_1'(t)y_1(t) + u_1(t)y_1'(t) + u_2'(t)y_2(t) + u_2(t)y_2'(t)$$
$$y_p' = u_1(t)y_1'(t) + u_2(t)y_2'(t)$$

Differentiating

$$y_p'' = (t) = u_1'(t)y_1'(t) + u_1(t)y_1''(t) + u_2'(t)y_2'(t) + u_2(t)y_2''(t)$$

Plugging this into the original equation and cancelling gives

$$u_1'y_1' + u_2'y_2' = g(t) \qquad \text{Condition 2}$$

Solving the system given by the two conditions gives

$$u_1' = -\frac{y_2g(t)}{y_1y_2' - y_2y_1'} \qquad u_2' = \frac{y_1g(t)}{y_1y_2' - y_2y_1'}$$

so

$$u_1 = -\int \frac{y_2g(t)}{y_1y_2' - y_2y_1'}dt \qquad u_2 = \int \frac{y_1g(t)}{y_1y_2' - y_2y_1'}dt$$

And so our particular solution is

$$y_p(t) = -y_1(t)\int \frac{y_2g(x)}{y_1y_2' - y_2y_1'}dx + y_2(t)\int \frac{y_1g(x)}{y_1y_2' - y_2y_1'}dx$$

So our general solution is

$$y = y_c(t) + y_p(t)$$

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1.2.2.3 Substitution The Bernoulli equation

$$y'(t) + p(x)y = q(x)y^n$$

Can be solved with the substitution $u = y^{1-n}$

$$u = (1-p)(p(x)u + q(x))$$

Which can then be solved with other methods.

1.2.3 Exactly Solvable Cases

First Order Linear Equations

$$y' + p(t)y = q(t)$$

The general solution is

$$y = \frac{1}{M(t)} \int_{t_0}^{t} q(u)M(u)du + \frac{C}{M(t)}$$

for $M(t) = e^{\int_{s_0}^{t} p(s)ds}$ and any constant C.

Linear Equations with Constant Coefficients

$$Ly = \sum_{j=0}^{n} a_j D^j y = 0$$

Solutions exist in the form

$$y(t) = e^{\lambda t}$$

where λ is a root of the characteristic polynomial

$$p(\lambda) = \sum_{j=0}^{n} a_j \lambda^j$$

If roots are repeated, the solutions associated with the same root must take on forms that are orthogonal to one another, such as

$$y(t)_1 = e^{\lambda t}\ ,\ y(t)_2 = te^{\lambda t}\ ,\ y(t)_3 = t^2 e^{\lambda t}$$

The pair of solutions associated with a pair of complex roots must be real, and so for a pair of roots $(\lambda \pm (\alpha + i\beta))$

$$y_1(t) = e^{\alpha t} cos(\beta t) \qquad y_2(t) = e^{\alpha t} sin(\beta t)$$

Euler Type Equations

$$Ly = \sum_{j=0}^{n} a_j (t-t_0)^j D^j y = 0$$

Solutions exist in the form

$$y(t) = (t-t_0)^\lambda \qquad t \neq t_0$$

Where λ can be found by using this solution form in the equation, which forms the **indicial equation**

$$\sum_{j=0}^{n} A_j t^j = 0$$

Where $A_n = a_n$ but the other coefficients depend on the nature of the ODE. If the indicial equation has...

- two roots, then
$$y(t) = c_1(t-t_0)^{\lambda_1} + c_2(t-t_0)^{\lambda_2}$$
- one root, then one must perform reduction of order. However, solutions typically have a solution that looks something like
$$y(t) = c_1(t-t_0)^{\lambda} + c_2(t-t_0)^{\lambda}\ln|t-t_0|$$
For higher algebraic muliplicities of the root, you will have additional solutions $\{(t-t_0)^{\lambda}(\ln|t-t_0|)^2, ..., (t-t_0)^{\lambda}(\ln|t-$
- a complex pair of roots $r = \lambda \pm i\omega$, one must solve for the real solutions. Typically you end up with a solution that looks something like
$$y(t) = c_1(t-t_0)^{\lambda}\cos(\omega\ln|t-t_0|) + c_2(t-t_0)^{\lambda}\sin(\omega\ln|t-t_0|)$$
For higher algebraic multiplicities you can solve for real valued solutions of the form
$$(t-t_0)^{\lambda}\cos(\omega\ln|t-t_0|)\ln|t-t_0|\,,(t-t_0)^{\lambda}\sin(\omega\ln|t-t_0|)\ln|t-t_0|\,, ...$$
$$..., (t-t_0)^{\lambda}\cos(\omega\ln|t-t_0|)(\ln|t-t_0|)^{m-1}, (t-t_0)^{\lambda}\sin(\omega\ln|t-t_0|)(\ln|t-t_0|)^{m-1}$$

1.2.3.1 Example

$$ax^2y'' + bxy' + cy = 0$$

Yields the indicial equation
$$a\lambda^2 + (b-a)\lambda + c = 0$$

Say $a = 1, b = -6, c = 10$. Then $\lambda_{1,2} = 2, 5$ and
$$y(t) = c_1x^2 + c_2x^5$$

Say $a = 1, b = -9, c = 25$. Then $\lambda = 5$ and we must additionally solve
$$y(x) = x^5u(x) \qquad v = u'$$

which has the solution
$$y(x) = x^5(c_1\ln|x| + c_2)$$

Say $a = 1, b = -3, c = 20$. Then $\lambda = 2 \pm 4i$
$$y(x) = c_1x^2\cos(4\ln|x|) + c_2x^2\sin(4\ln|x|)$$

1.2.4 Relation between Euler Equations and Constant Coefficient Equations

Let $y : (t_0, \infty) \to \mathbb{R}$ for and $Y : (-\infty, \infty) \to \mathbb{R}$ be functions of t and x respectively. Assume they are related by a substitution $x = e^t$. That is, $y(t) = Y(x)$. Then the Euler equation for y can be related to the constant coefficient equation for Y.

1.3 Nonlinear Equations

Nonlinear equations such as
$$y' = y^2 \text{ with } u(t_0) = u_0$$

May have a unique solution, but usually only local solution.

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1.4 Nonlinear ODEs

Nonlinear ODEs hav the form

$$F(t, y, y', ..., y^{(n)}) = 0$$

where F depends nonlinearly on $y^{(n)}$. It may have an explicit form of

$$y^{(n)} = f(t, y', y'', .., y^{(n-1)})$$
$$y(a) = \gamma_0$$
$$\vdots$$
$$y^{(n-1)}(a) = \gamma_{n-1}$$

2 Solutions

2.1 General Solutions

General Solutions are the set of all solutions to a DE. Generally, a n^{th} order D's general solution has n arbitrary constants.

Normalized Solutions: The solution set (for example $y(x) = c_1 y_1(x) + c_2 y_2(x)$ to a DE such that when $y(x = 0) = 0$ and $y'(x = 0) = 1$.

2.1.1 Well-Posed Problems

A problem is well posed if

- There is one solution (existence)
- The solution is unique (uniqueness)
- The solution depends continuously on the data (stability condition)
  Small changes in the intial or boundary conditions lead to small changes in the solution

Wronksian: The determinant of the Fundamental Matrix of a set of solutions to a differential equation. A set of solutions to a DE are linearly independent if the Wronskian identically vanishes for all $x \in I$. Note that $W \equiv 0$ does not imply linear dependence.

For f, g, $W(f, g) = fg' - gf'$. For n real or complex valued functions $f_1, f_2, ..., f_n$ which are $n - 1$ times differentiable on an interval I, the Wronksian $W(f_1, ..., f_n)$ as a function on I is defined by

$$W(f_1, ..., f_n)(x) = \begin{vmatrix} f_1(x) & \dots & f_n(x) \\ f_1'(x) & \dots & f_n'(x) \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{vmatrix} x \in I$$

3 Advanced Theory

3.1 First Order Equations

Consider

$$\vec{\mathbf{x}}' = f(\vec{\mathbf{x}})$$

3.1.1 Intervals of existence

Let $F(x) = \int_{x_0}^{x} \frac{dy}{f(y)}$. If $x = \phi(t)$ is a solution to the ODE, then $F(\phi(t)) = t$ so $F(\phi(0)) = 0$. By the inverse mapping theorem,

$$F(x) = t + t_0 \implies x = F^{-1}(t + t_0) \implies x = F^{-1}(t + F(x_0))$$

Consider two cases

- If $f(x_0) = 0$ then
$$\phi(t) = x_0 \; \forall \; t, u(t) = f(x_0) = f(\phi(t))$$
- If $f(x_0) \neq 0$ then $f \in C(\mathbb{R}) \implies f \neq 0$ in some neighborhood about x_0. Assuming $f(x) > 0$ on (a, b), then $F'(x) = \frac{1}{f(x)} > 0$, $a < x < b$ implies $F(x)$ is monotone increasing so $F^{-1}(x)$ exists and $\phi(t) = F^{-1}(t)$ is a solution.

4 New Notes

Basic Existence and Uniqueness

Let $U \in \mathbb{R}^{n+1}$ be open, $f \in C(u)$, and $(t_0, x_0) \in U$. If f satisfies a Lipschitz condition in x uniformly in t on some closed spacetime cylinder $\mathcal{S}$ that is contained in U, then there is an interval $[t_0, t_0 + T_0] \subset [t_0, t_0 + T]$ and a unique solution of $x' = f(x, t), x(t_0) = x_0$. Picard iteration converges uniformly to some $\varphi(t)$ that satisfies the IE

$$\varphi(t) = \lim_{k \to \infty} \varphi_{k+1}(t) = \lim_{k \to \infty} \left(x_0 + \int_{t_0}^{t} f(s, \phi_k(s)) ds \right) = x_0 +_{t_0}^{t} f(s, \varphi(s)) ds$$

Suppose $\phi(t), \psi(t)$ satisfy

$$\phi(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s)) ds, \psi(t) = x_1 + \int_{t_0}^{t} f(s, \psi(s)) ds$$

for $t \in [t_0, t_1]$, then

$$\|\phi(t) - \psi(t)\|_C \leq \|x_0 - x_1\| \; + \int_{t_0}^{t} L \|\phi(s) - \psi(s)\| \; ds$$

Satisfies Gronwall's inequality with $u(t) = \|\phi(t) - \psi(t)\|_C$, $u_0 = \|x_0 - x_1\|$, and $v(s) = 1$. Thus

$$\|\phi(t) - \psi(t)\|_C \leq \|x_0 - x_1\| \; e^{L(t - t_0)} \implies \max_{t_0 \leq t \leq t_1} \|\phi(t) - \psi(t)\|_C \leq \|x_0 - x_1\| \; e^{L(t_1 - t_0)}$$

So $x_1 = x_0 \implies \phi(t) = \psi(t)$, and the solution depends continuously on the initial data.

Let $\phi(t)$ be the solution of $x' = f(t, x), x(t_0) = x_0$ and let $\psi(t)$ be the solution of $x' = f(t, x), x(t_1) = x_1$ Suppose both solutions exist of a common interval (a, b) with $t_0, t_1 \in (a, b)$. We know

$$\phi(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s)) ds, \psi(t) = x_1 + \int_{t_1}^{t} f(s, \phi(s)) ds, a < t < b$$

Without loss of generality, assume $t_0 < t_1$. So

$$\begin{aligned}
\phi(t) - \psi(t) &= x_0 - x_1 + \int_{t_0}^{t} f(s, \phi(s) ds - \int_{t_1}^{t} f(s, \psi(s)) ds \\
&= x_0 - x_1 + \int_{t_0}^{t_1} f(s, \phi(s) ds + \int_{t_1}^{t} f(s, \phi(s)) - f(s, \psi(s)) ds \\
\|\phi(t) - \psi(t)\|_C &= \|x_0 - x_1\| \; + |t_1 - t_0| \, \|f(s, \phi(s)\|_C + L(t - t_1) \|f(s, \phi(s)) - f(s, \psi(s))\|_C
\end{aligned}$$

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So letting $\mu = \|x_0 - x_1\| \; + |t_1 - t_0| \left\| f(s, \phi(s) \right\|_C$, and $V(s) = L(t - t_1)$, we use Gronwall's to obtain

$$\|\phi(t) - \psi(t)\|_C \leq (\|x_0 - x_1\| \; + |t_1 - t_0| \left\| f(s, \phi(s) \right\|_C) \, e^{L(t-t_0)}, t > t_1$$

So $x = \phi(t) = \phi(t, t_0, x_0)$. That is, ϕ is a continuous function of the problem parameters as well as t.

Differentiation on $\mathbb{R}^n$ Given $F : \mathbb{R}^N \to \mathbb{R}^N$, we say $F(x)$ is differentiable at $\vec{\mathbf{x}}_0$ if there exists a linear map (matrix) A such that

$$\lim_{h \to 0} \frac{\|F(x_0 + h) + F(x_0) - Ah\|}{\|h\|} = 0$$

We denote $DF(x_0) = A = \frac{\partial F_i}{\partial x_j}(\vec{\mathbf{x}}_0)$ if $F = (F_1, ..., F_N)^T$.
F is differentiable at $\mathbb{R}^N$ If F is differentiable for all $\vec{\mathbf{x}}_0 \in \mathbb{R}^N$ then $DF(\vec{\mathbf{x}})$ is a matrix valued function.

$F \in C^1(\mathbb{R}^N)$ if $x \to DF(\vec{\mathbf{x}})$ is continuous with respect to some norm ???

$f \in C(U) \implies D_x f \in C(U, \mathbb{R}^{N \times N})$ f is locally Lipschitz continuous in x with respect to t. That is, given a compact subset $U_0 \subset U$, there is a constant $L > 0$ such that

$$\|f(t, x) - f(t, y)\| \; \leq L_{U_0} \|x - y\| \; , \; \forall \; (t, x), (t, y) \in U_0$$

We assume $U_0 = [\alpha, \beta] \times K$ where K is compact and convex. Let $x, y \in K$ and $s \in (0, 1)$ and define

$$F(s) = f(t, x + s(y - x)), 0 \leq s \leq 1$$

By the Chain Rule,

$$F'(s) = \frac{d}{ds} f(t, x + s(y - x)) = [f_x(t, x + s(y - x))]_{N \times N} \cdot [(y - x)] +_{N \times 1}$$

Now consider $F(0) = f(t, x)$, $F(1) = f(t, y)$. So

$$F(1) - F(0) = \int_0^1 F'(s) ds \iff f(t, y) - f(t, x) = \int_0^1 [f_x(t, x + s(y - x))]_{N \times N} \cdot [(y - x)] +_{N \times 1} ds$$

so let $L = \| \| \| f_x(t, x + s(y - x)) \| \; \|_C$

$$\|f(t, y) - f(t, x)\|_C \leq \int_0^1 L \|y - x\| \; ds \leq L \|y - x\|$$

Example: $y' = y^2 = f(y)$

$$f(x) - f(y) = f'(s)(x - y), y \leq s \leq x$$

We see $f'(y) = 2y$, so these intervals all have different constants.

4.1 The First Variational Equation

Consider IVP1 $x' = f(t, x)$ $x(t_0) = x_0$. If $f \in C^1(U)$ then $x = \phi(t, \tau, \xi)$ is differentiable in all three variables.

$$\frac{\partial}{\partial t} \frac{\partial}{\partial \tau} \phi = f_x(t, \phi) \cdot \frac{\partial}{\partial \tau} \phi \implies y(t) = \frac{\partial}{\partial \tau} \phi(t, \tau, \xi) \text{ solves } \frac{dy}{dt} = A(t) y$$

We call $\frac{dy}{dt} = A(t)y$ the First Variational Equation, where $A(t) = f_x(t, x, \tau, \xi)$.
Performing the same thing for $\frac{\partial}{\partial \xi}$

$$\frac{dX}{dt} = A(t) X$$

So $\phi(t, \tau, \xi)$ satisfies

$$\phi(\tau, \tau, \xi) = \xi \implies \frac{\partial}{\partial \xi} \phi(\tau, \tau, \xi) = I$$

Also,

$$\phi(\tau,\tau,\xi)=\xi \implies \frac{\partial}{\partial t}\phi+\frac{\partial}{\partial \tau}=0 \implies \frac{\partial}{\partial \tau}\phi=-f(\tau,\xi)$$

$\phi(t,\tau,\xi)$ has partial derivatives determined by

$$\frac{\partial}{\partial \tau}\phi : \begin{cases} y'=A(t)y \\ y(\tau)=-f(\tau,\xi)\end{cases} \qquad \frac{\partial}{\partial \xi}\phi : \begin{cases} X'=A(t)X \\ X(0)=I\end{cases}$$

We want to show

$$\lim_{h\to 0}\frac{\|Q(t,\tau,\xi,h)\|}{\|h\|}=0, \qquad Q(t,\tau,\xi,h)=\phi(t,\tau,\xi+h)-\varphi(t,\tau,\xi)-X(t)h$$

Let $(\tau,\xi)\in B(t_0,x_0;a_1,b_1)$ and choose h sufficiently small $h\in\mathbb{R}^n$ so that $(\tau,\xi+h)\subset \mathbb{R}_1=\overline{B(t_0,x_0;a,b)}$. We use

$$\phi(t)=\phi(t,\tau,\xi) \qquad \phi_h(t)=\phi(t,\tau,\xi+h) \qquad A(t)=f_x(t,\phi(t,\tau,\xi))$$

These are all defined on $[\tau-T,\tau+T]$ with $(t,\phi(t))\in\mathbb{R}^2$ and $(t,\phi_h(t))\in\mathbb{R}^2$ for $t\in[\tau-T,\tau+T]$. Using $L=\|\|\|f_x(t,x)\|_1\|_C$ and $\|A(t)\| \ \le L$. $t\in[\tau-T,\tau+T]$.

$$\|\phi(t)-\phi_h(t)\| \ \le \|h\| \ e^{2LT}$$

$$\phi(t)=\xi+\int_\tau^t f(s,\phi(s))ds$$

$$\phi_h(t)=\xi+h+\int_\tau^t f(s,\phi_h(s))ds$$

$$X(t)=I+\int_\tau^t A(s)X(s)ds$$

So

$$Q(t,\tau,\xi,h)=\int_\tau^t \left(f(s,\phi_h(s))-f(s,\phi(s))-A(s)X(s)h\right)ds$$

Using $f(t,x)-f(t,y)=\int_0^1 f_x(t,x+\sigma(y-x))\cdot(y-x)d\sigma$,

$$\|f(t,x)-f(t,y)-f_x(t,x)\cdot(y-x)\| \ \le \int_0^1 \|f_x(t,x+\sigma(y-x)))-f_x(t,x)\| \ \|y-x\| \ d\sigma$$

Since $f_x(t,x)$ is continuous on the compact set R_2, it is uniformly continuous on R_2. It follows that $f_x(t,x)$ is uniformly continuous on R_2.

We seek to use Gronwall's inequality. We see that

$$Q(t,\tau,\xi,h)=\int_\tau^t \left(f(s,\phi_h(s))-f(s,\phi(s))-A(s)(\phi_h(s)-\phi(s))\right)ds+\int_\tau^t A(s)Q(t,\tau,\xi,h)ds$$

Continuity of f allows us to pick $\epsilon_1<\frac{\epsilon}{Te^{3LT}}$ so that

$$\|Q\| \ \le \int_\tau^t \epsilon_1\|\phi_h-\phi\| \ ds+\int_\tau^t L\|Q\| \ ds\le \epsilon_1 T\|h\| \ e^{2LT}+\int_\tau^t L\|Q\| \ ds$$

Using Gronwall's inequality, we get

$$\|Q\| \ \le \epsilon_1 T\|h\| \ e^{2LT}e^{TL}<\epsilon$$

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4.1.0.1 Differentiability With Respect to a Parameter Consider

$$\begin{cases} x' = f(t,x,\lambda) & x \in \mathbb{R}^N \\ x(\tau) = \xi \end{cases}$$

$$f : U \times \Lambda \to \mathbb{R}^N, U \subset \mathbb{R}^{N+1}, \Lambda \subset \mathbb{R}^P, (\tau,\xi) \in U$$

Let $f \in C^1(U \times \Lambda)$. Solutions of IVP are functions $\vec{\mathbf{x}} = \phi(t,\tau,\xi,\lambda)$.
So let $\vec{\mathbf{y}}' = 0, \vec{\mathbf{y}}(\tau) = \lambda$, and we recast our problem as

$$\vec{\mathbf{z}} = \begin{pmatrix} \vec{\mathbf{x}} \\ \vec{\mathbf{y}} \end{pmatrix}_{(n+p)\times 1}, \qquad \vec{\mathbf{z}}' = \begin{pmatrix} \vec{\mathbf{x}}' \\ \vec{\mathbf{y}}' \end{pmatrix} = \begin{pmatrix} f(t,z) \\ 0 \end{pmatrix} = F(t,z), \qquad \vec{\mathbf{z}}(\tau) = \begin{pmatrix} \vec{\tau} \\ \vec{\lambda} \end{pmatrix} = \gamma \implies z = \psi(t,\tau,\gamma)$$

So if we have

$$a(t) = \begin{pmatrix} \frac{\partial}{\partial x} f(t,\psi(t,\tau,\gamma)) & \frac{\partial}{\partial \lambda} f(t,\psi(t,\tau,\gamma)) \\ 0_{p\times n} & 0_{p\times p} \end{pmatrix}$$

Then our matrix DE is

$$Z' = a(t)Z, Z(0) = I \implies Z = \frac{\partial}{\partial \gamma}\psi = \begin{pmatrix} \frac{\partial}{\partial \xi} x & \frac{\partial}{\partial \lambda} x \\ \frac{\partial}{\partial \xi} y & \frac{\partial}{\partial \lambda} y \end{pmatrix} = \begin{pmatrix} x_\xi & x_\lambda \\ 0 & I_p \end{pmatrix}$$

That is,

$$\begin{pmatrix} x_\xi & x_\lambda \\ 0 & I_p \end{pmatrix}' = \begin{pmatrix} f_x & f_\lambda \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_\xi & x_\lambda \\ 0 & I_p \end{pmatrix} \iff x_\xi' = f_x x_\xi, x_\lambda' = f_x x_\lambda + f_\lambda$$

With initial conditions $x_\xi(\tau) = I$, $x_\lambda(\tau) = 0$.

4.2 Continuation of Solutions

Theorem. If $f \in C^1(U)$ and $x = \phi(t)$ is a solution of our above IVP, defined on an open interval (a,b) containing t_0. If

$$\{(t,\phi(t)) : a < t < b\} \subset K \subset U$$

for some compact K then $\phi(a^+) = \lim_{t\to a^+} \phi(t)$ and $\phi(b^-)$ both exist as finite values with $(a,\phi(a^+)),(b,\phi(b^-)) \in U$. Hence $\phi(t)$ extends as a continuous function to $[a,b]$ which can be further extended as a solution of $x' = f(t,x)$ to a larger interval.
Proof. Let $M = \max|f(t,x)| : (t,x) \in K$. We know

$$\phi(t_2) - \phi(t_1) = \int_{t_1}^{t_2} f(s,\phi(s))ds, a < t_1 < t_2 < b$$

so $\|\phi(t_2) - \phi(t_1)\|_C \leq M|t_2 - t_1|$. Thus $\phi(t)$ is uniformly continuous thus $t_n \to b$ produces a Cauchy sequence $\phi(t_n)$ which converges by completeness to $\hat{\phi}(b)$
If s_n us another sequence such that $s_n \to b$,

$$\left\|\hat{\phi}(b) - \phi(s_n)\right\|_C \leq \left\|\hat{\phi}(b) - \phi(t_n)\right\|_C + \|\phi(t_n) - \phi(s_n)\|_C \to 0$$

Thus the limit is unique.
Corollary. If $x = \phi(t)$ is a solution on (a,b) and $\phi(t)$ cannot be extended beyond b as a solution, then $(t,\phi(t))$ must leave every compact subset of U as $t \to b^+$. Analogous statements holds at the left endpoint a.

4.2.1 Extension of Solutions

Suppose $\Omega = (a,b) \times \mathbb{R}^N$, $f \in C(\Omega), -\infty \leq a < b \leq \infty$ and satisfies a Lipschitz condition in x, uniformly in t

$$\|f(x,t) - f(t,y)\| \leq L\|x-y\| , \forall (t,x),(t,y) \in \Omega$$

Then solutions of $x' = f(t,x)$ exist on the entire interval (a,b). Notice that for all $t \in [a,b] \subset (a,b)$ and $x \in \mathbb{R}^N$,

$$\|f(t,x)\| = \|f(t,0)\| + \|f(t,x) - f(t,0)\| \leq \max_{t\in[a,b]} \|f(t,0)\| + L\|x\| = M_{\alpha,\beta} + L\|x\|$$

So for any τ, t such that $\alpha \leq \tau \leq t \leq \beta$ implies

$$\|\phi(t)\| \leq \|\phi(\tau)\| + \int_\tau^t \|f(s,\phi(s))\| \, ds \leq \|\phi(\tau)\| + M_{\alpha,\beta}(\beta-\alpha) + \int_\tau^t L\|\phi(s)\| \, ds$$

Thus $\|\phi(t)\|$ is bounded on $[\alpha,\beta]$ for any compact subinterval of (a,b). That is $(t,\phi(t))$ cannot leave every compact subset of Ω on any interval of the form $[\alpha,\beta] \subset (a,b)$ so the solutions extend to (a,b)

4.3 Existence

$$x' = f(t,x), f \in C(\Omega) \qquad x(\tau) = \xi$$

Consider the space-time cylinder $R = \overline{B}(\tau, \xi : a, b) \subset \Omega$. Let $M = \max_{(t,x)\in R} \|f(t,x)\|$. $\alpha = \min\left\{a, \frac{b}{M}\right\}$. Choose a partition $\{t_j\}_{j=0}^N$ of $[\tau, \tau+\alpha]$.

$$\tau = t_0 < t_1 < ... < t_{N-1} < t_N = \tau + \alpha$$

Define an approximate solution $\phi(t)$ by

$$\phi(t_{(j+1)}) = \phi(t_j)_f(t_j, \phi(t_j))(t_{j+1} - t_j), j = 0, ..., N-1$$

with $\phi(t_0) = \xi$. Use linear interpolation to get

$$\phi(t) = \phi(t_j) + f(t_j, \phi(t_j))(t - t_j), t \in [t_{j+1}, t_j]$$

Notice, $\phi(t)$ is continuous, but not differentiable at the nodes $\{t_j\}$

5 Linear Systems

$$\begin{cases} \vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + g(t) \\ \vec{\mathbf{x}}(\tau) = \xi \end{cases}$$

where $A \in \mathbb{R}^{n\times n}$, $g \in \mathbb{R}^n$. $A \in C((a,b), \mathbb{R}^{n\times n})$ and $g \in C((a,b), \mathbb{R}^n)$. There is a unique solution of the IVP for every $(\tau, \xi) \in (a,b) \in \mathbb{R}^n$ that is valid on (a,b). Under the above assumption, the set of solutions to the homogenous problem $(g(t) \equiv 0)$ is an n-dimensional linear space. So if $\vec{\phi}(t)$ is the solution to $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$, $\vec{\mathbf{x}}(\tau) = \vec{\mathbf{e}}_i$, then $\vec{\psi}(t) = \sum_{i=1}^n \xi_i \vec{\phi}_i(t)$ is in the $\operatorname{span}\left\{\vec{\phi}_1(t), ..., \vec{\phi}_n(t)\right\}$. Thus we have a basis for our solution space. Denoting $\Phi(t) = (\vec{\phi}_1(t), ..., \vec{\phi}_n(t))$, then

$$\Phi'(t) = A(t)\Phi(t)$$

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5.1 Determinants

Properties

- d

5.2 The Homogenous Case

$$\begin{cases} \vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} \\ \vec{\mathbf{x}}(\tau) = \xi \end{cases}$$

5.2.0.1 Superposition Principle

If $x_1(t), x_2(t)$ are solutions then so is $x_3(t) = c_1x_1(t) + c_2x_2(t)$ for any c_1, c_2.

5.2.0.2 General Solution

A **fundamental set of solutions** is a set $\{x_i(t)\}_i$ such that they form a linearly independent set of solutions. Then $x(t) = c_1x_1(t) + ...c_nx_n(t)$ is a general solution (all solutions can be written in this form). The **Fundamental Matrix** is

$$X(t) = (x_1(t), ..., x_n(t))_{n\times n}$$

Note $\det[X(t)] \neq 0$, and any solution $x(t)$ can be expressed as $X(t)\vec{\mathbf{c}}$ for some $\vec{\mathbf{c}}$. This means any $Y(t) = X(t)\vec{\mathbf{c}}$ is also a fundamental matrix with $\det[Y(t)] = \det[X(t)] \neq 0$.

- If $X(t)$ and $Y(t)$ are fundamental matrices then there exists a nonsingular C such that $Y(t) = X(t)C$. In fact, $C = X^{-1}(t)Y(t)$.

5.3 The Inhomogenous Case

$$\begin{cases} \vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + g(t) \\ \vec{\mathbf{x}}(\tau) = \xi \end{cases}$$

- If ψ_1, ψ_2 are two solutions, then $\psi = c(\psi_1 - \psi_2)$ is a solution of the homogenous case for any c. That is if we know the fundamental solution set for the homogenous case, we can simply add on a particular solution to the inhomogenous problem.

5.3.0.3 Variation of parameters

Let $X(t)$ be a fundamental matrix. Consider $x(t) = X(t)\vec{\mathbf{c}}$.

$$x'(t) = X'(t)\mathbf{c}\vec{(\mathbf{t})} + X(t)\mathbf{c}'\vec{(\mathbf{t})} = A(t)\mathbf{x}\vec{(\mathbf{t})} + X(t)\mathbf{c}'\vec{(\mathbf{t})} = A(t)x(t) + g(t)$$

So let $\mathbf{c}'\vec{(\mathbf{t})} = X^{-1}(t)g(t)$ since $\det[X(t)] \neq 0$ for all t. So we have a **particular solution**

$$x_p(t) = X(t)\mathbf{c}\vec{(\mathbf{t})} = X(t)\int_\tau^t X^{-1}(s)g(s)ds, \tau \in (a,b)$$

Note $x_p(t) = 0$. So any solution of the inhomogenous problem can be written in the form

$$x(t) = X(t)\vec{\mathbf{c}} + X(t)\int_\tau^t X(s)g(s)ds$$

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However if we consider the IVP with $x(\tau)=\xi$, then pick $\vec{\mathbf{c}}=X^{-1}(\tau)\xi$ so

$$x(t)=X(t)X^{-1}(\tau)\xi+X(t)\int_\tau^t X(s)g(s)ds$$

and we see

$$x(\tau)=X(\tau)X^{-1}(\tau)\xi+X(\tau)\int_\tau^\tau X(s)g(s)ds=\xi$$

5.3.0.4 State Transition Matrix

Now denote the State Transition Matrix $\Phi(t,\tau)=X(t)X^{-1}(\tau)$ and we see since

$$x(t)=\Phi(t,\tau)\xi+\int_\tau^t \Phi(t,\tau)g(s)ds$$

and thus $\Phi(t,\tau)$ solves the problem

$$\begin{cases}\vec{\mathbf{X}}'=A(t)X\\ \vec{\mathbf{X}}(\tau)=I\end{cases}$$

By construction, $Phi(t)$ is uniquely determined.

5.4 Special Case: Constant Coefficient System

If $A(t)=A$ is independent of t, then any solution $\phi(t)$ is still a solution when translated so you can take the initial time to be 0 via the translation $\phi(t-\tau)$. So then $\Phi(t,\tau)=\Phi(t-\tau)$. So we have

$$\begin{cases}\Phi'=A\Phi\\ \Phi(0)=I\end{cases}$$

Thus the solution is $\Phi(t)=e^{At}$, where the matrix exponential can be defined in one of three ways:

- $e^{At}=\sum\limits_{n=0}^{\infty}\frac{t^n}{n!}A^n$
- $X(t)=e^{At}$ satisfies $X'=AX$, $X(0)=I$
- Use an eigendecomposition $\Lambda=V^{-1}AV$ to get $e^{tA}=Ve^{t\Lambda}V^{-1}$

We note the properties of the matrix exponential

- $\frac{d}{dt}e^{tA}=Ae^{tA}$ and $e^{(0)A}=I$
- $Ae^{tA}=e^{tA}A$ for all $t\in\mathbb{R}$
- If $AB=BA$, then $e^Ae^B=e^{A+B}$
- $e^{t_1A}e^{t_2A}=e^{(t_1+t_2)A}$ for all t_1,t_2
- $\left(e^{tA}\right)^{-1}=e^{-tA}$ for all t
- $\det\left(e^{tA}\right)=e^{t\,\mathrm{tr}(A)}$ for all t (Abel's Formula)
- If B is nonsingular, $B^{-1}e^{tA}B=e^{tB^{-1}AB}$

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5.5 2D Constant Coefficients Case

$$\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t), A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Let $\vec{\mathbf{x}} = \vec{\phi}(t, x_0)$ be the solution satisfying $\vec{\mathbf{x}}(a) = x_0$

5.5.1 Invertible Matrix Case

If $\det A = 0$, then the only **rest point** is $\vec{\mathbf{x}} = \vec{\mathbf{0}}$. If $T = a + d$ and $D = ad - bc$, then $\lambda_{\pm} = \frac{1}{2}\left(T \pm \sqrt{T^2 - 4D}\right)$, and we consider various cases and their subcases.

- $T^2 - 4D > 0$
  - $\lambda_- > 0, \lambda_+ > 0$ $(T > 0, D > 0)$ Moving along parabolas away from the oringin
  - $\lambda_- < 0, \lambda_+ < 0$ $(T < 0, D > 0)$ Moving along parabolas toward the origin
  - $\lambda_- < 0 < \lambda_+ < 0$ $(D < 0)$ Mixed behavior
- $T^2 - 4D = 0$
  - $e^{tJ} = e^{\lambda t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
  - $e^{tJ} = e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$
- $T^2 - 4D > 0$ We have $e^{At} r \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} = re^{\alpha t} \begin{pmatrix} \cos(\theta + \beta t) \\ \sin(\theta + \beta t) \end{pmatrix}$
  - $T > 0$ spirals outward from the origin
  - $T = 0$ rotates about the origin at fixed radius
  - $T < 0$ spirals inward toward the origin

5.6 Periodic Linear Systems

Consider

$$x' = A(t)x, A(t + T) = A(t)$$

5.6.0.1 Floquet Theory Let $A(t)$ be an $n \times n$ continuous T-periodic matrix.

- If $\Phi(t)$ is a fundamental matrix then so is $\Phi(t + \tau)C$ for any nonsingular constant matrix C.
- If $\Phi(t)$ is a fundamental matrix then there is a nonsingular $T-$periodic matrix $P(t)$ and a constant matrix R such that $\Phi(t) = P(t)e^{tR}$

Proof: (1) If $\Psi = \Phi(t+\tau)$, then $\Psi'(t) = \Phi'(t) = A(t+\tau)\Phi(t+\tau) = A(t)\Psi(t)$ and $\det(\Psi(t)) = \det(\Phi(t+\tau)) \neq 0$. (2) Since $\Phi(t)$ and $\Phi(t + \tau)$ are both fundamental matrices there is a nonsingular C such that $\Phi(t + \tau) = \Phi(t)C$, with $C = \Phi^{-1}(0)\Phi(t)$. Since C is nonsingular, C has a logarithm. Let $R = \frac{1}{T}\log(C)$ so that $e^{TR} = e^{\log C} = C$. Now define $P(t) = e^{-tR}$ so that $\Phi(t) = P(t)e^{tR}$. Note that $P(t+T) = \Phi(t+T)e^{-(t+T)R} = \Phi(t)Ce^{-TR}e^{-tR} = \Phi(t)CC^{-1}e^{-tR} = P(t)$.

5.6.1 Monodromy Matrix

There exits a C such that $X(t+\tau) = X(t)C$. Let $R = \frac{1}{T}\log(C)$. This implies

$$X(t) = P(t)e^{tR}, \Phi(t,t_0) = X(t)X^{-1}(t_0) \iff \begin{cases} \Phi' = A(t)\Phi \\ \Phi(t_0,t_0) = I \end{cases}$$

The Monodromy Matrix is

$$M(t_0) = \Phi(t_0+T,t_0) = X(t_0+T)X^{-1}(t_0) = P(t_0+T)e^{(t_0+T)R}e^{-t_0R}P^{-1}(t_0) = P(t_0)e^{TR}e^{-1(t_0)}$$

If $X(t_0) = I$, then $P(t_0) = I$ so $M = e^{TR}$. Also, if $X(0) = I$, then $X(T) = X(0)M = M$.

5.6.2 Invariants for Periodic Systems

Let $X(t) = P(t)e^{tR}$ as before, and suppose $Y(t)$ is another fundamental matrix. Then we know there are constant nonsingular matrices $B, \hat{C}$ such that

$$Y(t) = X(t)B, \qquad Y(t+T) = Y(t)\hat{C}$$

We have

$$Y(t+T) = X(t+T)B = X(t)CB = X(t)e^{TR}B = Y(t)B^{-1}e^{TR}B \implies \hat{C} = B^{-1}e^{TR}B = e^{TB^{-1}RB}$$

So then using previous information, $B^{-1}TRB = \frac{1}{T}\log(\hat{C})$ so then we see $Q(t) = Y(t)e^{-tB^{-1}RB}$ is $T-$periodic. Thus,

- Any fundamental matrix $Y(t)$ as the form

  $$Y(t) = Q(t)e^{tS}, \qquad S = B^{-1}RB$$

  where $Q(t)$ is nonsingular and $T-$periodic and S is a constant matrix that is unquely determined up to similarity transforms (and branches of the logarithm).

- The eigenvalues of S are the characteristic exponents

- The eigenvalues of e^{TS} are called the Floquet multipliers. Note that if λ is an eigenvalue of S then $\rho = e^{T\lambda}$ is an eigenvalue of e^{TS}. If $Re(\lambda < 0$, then $|\rho| < 1$.

Assuming $X(0) = I$, $P(0) = I$.

$$X(t) = Pe^{tR}\eta \implies X(nT) = P(nT)e^{nTR}\eta = \left(e^{TR}\right)^n \eta$$

Let $\eta = \alpha_1 y_1 + \alpha_2 y_2 + ...$ be the eigenvectors of e^{TR}. So then

$$X(t) = \alpha_1 \left(e^{TR}\right)^n y_1 + ... = \alpha_1 \rho_1^n y_1 + ...$$

So we see if $|\rho_i| < 1$ for all i, then $\lim\limits_{n\to\infty} X(nT) = 0$ since

$$\|X(t)\| \ \leq K\|x(nT)\|$$

So $X(0) = I$ implies $M = X(t)$ and the eigenvalues of M are the Floquet multipliers.

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6 Dynamical Systems

Consider autonomous systems

$$x' = f(x), f : \Omega \to \mathbb{R}^N, f \in C^1(\Omega), x(\tau) = \xi \in \Omega$$

$\phi(t, \tau, \xi)$ denotes the unique solution. Let $\psi(t) = \phi(t - \tau, 0, \xi)$ then

$$\psi'(t - \tau, 0, \xi) = f(\phi(t - \tau, 0, \xi)) = f(\psi(t)), \psi(\tau) = \xi$$

This implies $\phi(t) = \phi(t - \tau 0, \xi) = \phi(t, \tau, \xi)$. So we let $\phi(t, \tau)$ denot ethe solution of

$$x' = f(x), x(0) = \xi$$

Terminology, The **orbit (or trajectory)** through ξ is the curve $\{(t, \phi(t, \xi)) : \alpha(\xi) < t < \beta(\xi)\}$ where $(\alpha(s), \beta(s))$ denotes the maximal interval of existence.
Example: $x' = x(1 - x)$ has solution $x = \frac{e^t}{e^t + C}$. and so for $\xi \neq 0$, $C = \frac{1}{\xi} - 1$ $(0 < \xi < 1)$. We see that $C > 0 \implies \alpha(s) = -\infty, \beta(s) = \infty$. $\xi < 0 \implies x(t)$ is defined on $(\log \left(1 - \frac{1}{\xi}\right), \infty)$. $\xi > 1 \implies x(t)$ is defined on $(-\infty, \log \left(1 - \frac{1}{\xi}\right))$.

6.1 Straightening the Flow of a Vector Field

$$\frac{dx}{dt} = f(x) \qquad f(x_0) \neq 0$$

If $f(x_0) \neq 0$, then there is a change of variables to y such that locally, $\frac{dy}{dt} f(x_0)$ everywhere along the plane perpendicular to $f(x_0)$. Let $f_0 = f(x_0)$, and its jth component is $f_{0j} = f_j(x_0)$. Consider $y = \xi + t f_0$, where $\xi \in P = \left\{\xi \in \mathbb{R}^N : (\xi - x_0)^T f_0\right\}$. This is an orthogonal decomposition of y since $y - x_0 = \xi - x_0 + t f_0$, where $\xi - x_0 \perp f_0$. This implies

$$t = \frac{(y - x_0)^T f_0}{\|f_0\|^2} = t(y), \qquad \xi = y - t(y) f_0$$

Now $x = \psi(y) = \phi(t, \xi)$ where $t = t(y), \xi = \xi(y)$, where $\phi(t, \xi)$ are defined by $x' = f(x), x(0) = \xi$. By the chain rule,

$$\frac{\partial \psi}{\partial y_j} = \frac{\partial \phi}{\partial t} \frac{\partial t}{\partial y_j} + \frac{\partial \phi}{\partial \xi} \frac{\partial \xi}{\partial y_j} = \frac{f_{0j}}{\|f_0\|^2} f(\phi(t, s)) + \phi_\xi(t, \xi) \left(e_j - \frac{f_{0j}}{\|f_0\|^2} f_0 \right)$$

$$\frac{\partial t}{\partial y_j} = \frac{\partial}{\partial y_j} \frac{(y - x_0)^T f_0}{\|f_0\|^2}, \qquad \frac{\partial \xi}{\partial y_j} = \frac{\partial}{\partial y_j}(y - t(y) f_0) = e_j - \frac{f_{0j}}{\|f_0\|^2} f_0$$

At $y = x_0$, we have $\phi(t(x_0), \xi(x_0)) = \phi(0, x_0) = x_0$. $\phi_\xi(t(x_0), \xi(x_0)) = \phi_\xi(0, \xi) = I$. This implies

$$\frac{\partial \psi}{\partial y_j}|_{y = x_0} = \frac{f_{0j}^2}{\|f_0\|^2} f(x_0) + I \left[e_j - \frac{f_{0j}}{\|f_0\|^2} f_0 \right]$$

By the inverse function theorem, the map $x = \psi(y)$ is locally invertible (a diffeomorphism). If $y \in P$, say $y = \xi \in P$, then $t = 0$ and $\xi = y$. This implies

$$\frac{\partial \psi}{\partial y}(\xi) = [e_1 + s_1(f - f_0) \qquad ... \qquad e_n + s_n(f - f_0)] = I + [s_1(f - f_0) \qquad ... \qquad s_n(f - f_0)] = I + (f - f_0) \frac{f_0^T}{\|f_0\|^2}$$

This is a rank one perturbation of the identity. This is analoguous to

$$\left(I + uv^T\right)^{-1} = I - uv^T(1 - \alpha + \alpha^2 - ...) = I - \frac{1}{1+\alpha}uv^T, \qquad \alpha = uv^T \neq -1$$

So for $x = \psi(y) \iff y = \psi^{-1}(x)$

$$f(x) = \frac{dx}{dt} = \frac{\partial \psi}{\partial y}\frac{dy}{dt} \implies \frac{dy}{dt} = \left(\frac{\partial \psi}{\partial y}\right)^{-1} f(x)$$

Along P,

$$\frac{dy}{dt} = \left(I + \frac{(f - f_0)}{\|f_0\|^2} f_0^T\right) f(\xi) = f - (f - f_0) f_0$$

implies $x = \phi(0, \xi) = \xi$

6.2 Group Properties

Consider the system

$$x' = f(x) \qquad x(0) = \xi \qquad f \in C^1(\Omega), \Omega \in \mathbb{R}^n, \Omega \leftrightarrow M, \qquad \text{solution: } x = \phi(t, \xi) \qquad STAR!$$

The semi-group or group property is: $\phi(t + s, \xi) = \phi(t, \phi(s, \xi))$/

$$\phi_t : \Omega \to \Omega \phi(t) = \phi_t \circ \phi_s, \phi_0 = \mathrm{id}$$

Consider $\phi : \mathbb{R} \times \Omega \to \Omega$ solutions all defined on $\mathbb{R}$.
Consider $x' = x(1 - x)$. Let $T(\xi) = \ln\left(1 - \frac{1}{\xi}\right)$

$$\alpha(\xi) = \begin{cases} -\infty & \xi \leq 1 \\ T(\xi) & \xi > 1 \end{cases}, \beta(\xi) = \begin{cases} T(\xi) & \xi < 0 \\ \infty & \xi \geq 0 \end{cases}$$

This defines a set $W = U(\alpha(\xi), \beta(\xi)) \times \{\xi\}, \xi \in \mathbb{R}$. $\Phi : W \to \mathbb{R}, (t, x) \to \phi(t, x)$

6.3 Properties of the Flow Generated by STAR

- $\phi(t + s, \xi) = \phi(t, \phi(s, \xi))$
- Orbits cannot intersect transversally (ie with different tangent directions)
  The trajectory of a solution is the curve $\{(t, \phi(t, s)) : t \in (\alpha(\xi), \beta(\xi))\} \subset \mathbb{R} \times \Omega$. The orbit of a solution is the curve $\{\phi(t, \xi) : t \in (\alpha(\xi), \beta(\xi))\} \subset \Omega$.
- If $\phi(t_1, \xi) = \phi(t_2, \xi)$ for some $t_1 \neq t_2$, then $\phi(t, \xi)$ is periodic. Assume $t_2 > t_1$ and set $\psi_1(t) = \phi(t + t_1, \xi)$, $\psi_2(t) = \phi(t + t_2, \xi)$. Then $\psi_1(0) = \psi_2(0)$ and $\psi_1'(t) = f(\psi_1(t))$, $\psi_2'(t) = f(\psi_2(t))$ so $\psi_1(t) = \psi_2(t)$:

  $$\phi(t + t_1, \xi) = \phi(t + t_2, \xi), \ \forall\, t$$

  Set $t' = t + t_1$,

  $$\phi(t', \xi) = \phi(t' + t_2 - t_1, \xi) = \phi(t' + T, \xi)$$

  So we have a period $t_2 - t_1$.

6.3.1 Terminology

- Ω is the phase space or state space
- A point x_0 such that $f(x_0) = 0$ is called a critical point or equillibrium point, rest pt, steady state, fixed pt
- A critical pt x_0 is said to be non-degenerate if there is a neighborhood of x_0 that does not contain any other critical points.
  Note if $Df(x_0)$ is non-singular, then x_0 is isolated by the inverse function theorem.
-

6.4 The Pendulum Equation

$$\begin{pmatrix} \theta \\ \theta' \end{pmatrix}' = \begin{pmatrix} \theta' \\ -\frac{g}{L}\sin(\theta) \end{pmatrix}$$

The rest points are $F(\theta, \theta') = 0 \iff \theta = n\pi, n \in \mathbb{Z}, \theta' = 0$. This equation can be seen as

$$\theta'' + \frac{g}{L}\sin(\theta) = 0 \qquad \iff \qquad (\theta')^2 + \frac{2g}{L}(1 - \cos(\theta)) = const$$

that is, we notice the energy $E(\theta, \theta') = (\theta')^2 + \frac{2g}{L}(1 - \cos(\theta))$ along an orbit is constant. So the orbits are the level curves of the energy. We notice that $E(\theta, \theta')$ is $2\pi-$periodic and symmetric about both axes.
Consider the solution through $(0, \theta'_0)$ where $\theta'_0 > 0$

$$E(\theta, \theta') = E(0, \theta'_0) = \theta'_0 \implies \theta' = \sqrt{\theta_0'^2 - \frac{2g}{L}(1 - \cos(\theta))}$$

We have three cases

- $0 < \theta_0'^2 < \frac{4g}{L}$, there is a $\theta \in (0, \pi)$ st $\theta' = 0$
- $\frac{4g}{L} < \theta_0'^2$, there is no such value. That is, $\theta' > 0$ always
- $\theta_0'^2 = \frac{4g}{L}$

$$http://dmpeli.math.mcmaster.ca/Matlab/CLLsoftware/Pendulum/Pendulum2.gif$$

6.5 Critical Points

A critical point x_0 is said to be **Lyapunov stable** if for any given $\epsilon > 0$ there is a $\delta > 0$ such that for all points $\xi \in B(x_0, \delta) = \{x \in \mathbb{R}^n : \|x - x_0\| < \delta\}$ the solution of

$$x' = f(x), x(0) = \xi, (\text{with solution } \phi(t, \xi)) \implies \|\phi(t, \xi) - x_0\| < \epsilon \; \forall \; t > 0$$

A critical point x_0 is said to be **asymptotically stable** if it is stable and there is a number $_0 > 0$ such that $\xi \in B(x_0, \delta_0)$ implies $\lim_{t \to \infty} \phi(t, \xi) = x_0$.

6.5.0.1 Theorem Consider $x' = Ax$. If the real part of the eigenvalues of A are all negative, then $x = 0$ is an isolated rest point that is asymptotically stable (in fact, exponentially stable) since $\|\phi(t, \xi)\| \leq Ke^{-\lambda_1 t}\|\xi\|$. So given ϵ, choose $\delta = \frac{\epsilon}{\delta}$ so

$$\|\phi(t, \xi)\| \leq Ke^{-\lambda_1 t}\delta \leq K\delta = \epsilon, t \geq 0$$

So $x = 0$ is stable. And $\|\phi(t, \xi)\| \to 0$ as $t \to 0$ so $\phi(t, \xi) \to 0$ as $t \to \infty$.

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6.5.0.2 The Principle of Linearized Stability Suppose $x_0 \in \Omega$ is a critical point of $f(x)$ ($f(x_0) = 0$). Let $A = f'(x_0)$ so that $a_{ij} = \frac{df_i}{dx_j}(x_0)$. If all of the eigenvalues of A satisfy $Re(\lambda) < 0$ then x_0 is an asymptotically stable rest point of $x' = f(x)$. Proof: We consider the variational system obtained by changing coordinates $y = x - x_0 = \phi(t) - x_0$. This satisfies

$$y' = x' = f(x) = f(y+x_0) = f(y+x_0) - f(x_0) = f'(x_0)y + h(y) = Ay + h(y) \qquad h(y) = f(y+x_0) - f(x_0) - f'(x_0)$$

Clearly $x = x_0$ is asymptotically stable if and only if $y = 0$ is an asymptotically stable rest point of $y = Ay + h(y)$. We see that there exist $K \geq 1$, $\alpha > 0$, st $\left\|e^{tA}\right\| \leq Ke^{-\alpha t}$. Let $\sigma > 0$ be chosen so that $\sigma < \frac{\alpha}{K}$. Since $f' \in C(\Omega)$ there is a $\delta_0 \in (0, \epsilon)$ such that $\|f'(x_0 + sy) - f'(x_0)\| < \sigma$ for $y \in B_{\delta_0}$. Choose $\delta \in (0, \delta_0 K^{-1})$ and consider the solution $y = \psi(t)$ of (2) satisfying $y_0 \in B_\delta$ observe that

$$h(y) = \int_0^1 f'(x_0 + sy)yds - f'(x_0)y = \int_0^1 (f'(x_0 + sy) - f'(x_0))yds \implies \|h(y)\| \leq \sigma\|y\|$$

We now write for some $b > 0$,

$$\psi(t) = e^{tA}y_0 + \int_0^t e^{(t-s)A}h(\psi(s))ds \qquad 0 \leq t \leq b$$

Since $\delta < \delta_0$, there is a $b > 0$, such that $\|\psi(t)\| < \delta_0$ so

$$\|\psi(t)\| \leq Ke^{-\alpha t}\delta + K\int_0^t e^{-\alpha(t-s)}\sigma\|\psi(s)\| \, ds$$

Using Gronwall's inequality,

$$e^{\alpha t}\|\psi(t)\| \leq K\delta e^{K\sigma t} \implies \|\psi(t)\| \leq K\delta e^{-(\alpha - K\sigma)t} < \delta_0 < \epsilon, \qquad 0 \leq t < b$$

Define $\beta = \sup\{b > 0 : \|\psi(t)\| < \delta_0, 0 \leq t < b\}$. We must have $\beta = \infty$ otherwise we reach a contradiction. If $\beta < \infty$ then we obtained, by the same argument as above, $\|\psi(t)\| \leq K\delta < \delta_0$, $0 \leq t < \beta$ then there is a $b \geq \beta$ st $\|\psi(t)\| < \delta_0$, $0 \leq t < b$. Therefore for an autonomous system

- $\|\psi(t)\| \leq K\delta < \delta_0 < \epsilon$, $t \geq 0$
- $\|\psi(t)\| \leq K\delta e^{-(\alpha - K\sigma)t}$ $t \geq 0$, so $\lim_{t\to\infty} \psi(t) = 0$=

6.5.1 Non-Autonomous Systems

$$x' = f(t, x)$$

For a solution $x = \phi(t)$, let $y = x - \phi(t)$

$$\frac{dy}{dt} = A(t)y + h(t, y)$$

$$h(t, x) = f(t, y + \phi(t)) - f(t, \phi(t)) - f_x(t, \phi(t))y$$

We want to show $\|h(t, y)\| \leq \sigma\|y\|$ for small $\sigma > 0$.

Let $V : \overline{B(r)} \to \mathbb{R}$ be continuous and positive definite. Then there are functions $\psi_1, \psi_2 : [0, r] \to [0, +\infty]$ such that $\psi_1(0) = \psi_2(0)$ and $\psi_1(\|x\|) \leq V(x) \leq \psi_2(\|x\|)$. These functions are continuous and strictly monotone increasing. Note ψ_1, ψ_2 have inverses with $\psi_i(\|x\|) = c \iff \|x\| = \psi_i^{-1}(c)$, hence their level surfaces are spheres.

Proof: Let $s \in [0, r]$ and define $m(s) = \min\{V(x) : s \leq \|x\| \leq r\}$, $M(s) = \max\{V(x) : \|x\| \leq s\}$. so $m(\|x\|) \leq V(x) \leq M(\|x\|)$, $x \in \overline{B(r)}$. Also, $m(0) = M(0)$, $m(s) > 0$, $M(s) > 0$ for $s > 0$ since V is positive definite. Also the functions are continuous. We show this for $m(s)$. Let $\epsilon > 0$. Since $\overline{B(r)}$ is compact $V : \overline{B(r)} \to \mathbb{R}$ is uniformly continuous. Hence there is a $\delta > 0$ such that $|V(x_1) - V(x_2)| < \epsilon$ provided $\|x_1 - x_2\| < \delta$, for all $x_1, x_2 \in \overline{B(r)}$. Consider $0 < s_1 < s_2 \leq r$, with $|s_1 - s_2| < \delta$. Since $\{x : s_2 \leq \|x\| \leq r\} \subset \{x : s_1 \leq \|x\| \leq r\}$, we have $m(s_1) \leq m(s_2)$. We want to show $m(s_2) - \epsilon \leq m(s_1)$. Suppose $x \in \{x : s_1 \leq \|x\| \leq s_2\}$ and set $z = s_2 \frac{x}{\|x\|}$. Then $\|z\| = s_2$ so $V(z) \geq m(s_2)$. Also,

$$z - x = s_2 \frac{x}{\|x\|} - \|x\| \frac{x}{\|x\|} = (s_2 - \|x\|)\frac{x}{\|x\|} \implies \|z - x\| = s_2 - \|x\| \leq s_2 - s_1 < \delta$$

so

$$|V(z) - V(x)| < \epsilon \implies V(x) > V(z) - \epsilon \geq m(s_2) - \epsilon$$

This shows $V(x) > m_2(s) - \epsilon$ for all $x \in \{x : s_1 \leq \|x\| \leq s_2\}$. Thus we can conclude

$$m(s_2) - \epsilon < m(s_1) \leq m(s_2) < m(s_2) + \epsilon$$

so then $|s_1 - s_2| < \delta$ implies $|m(s_1) - m(s_2)| < \epsilon$. We have $m(s), M(s)$ satisfying all of the required properies except for the strictly monotone increasing property. To arrange for this we define $\psi_1(s) = \frac{s}{r} m(s)$ and $\psi_2(s) = (s+1)M(s)$ so these are strictly increasing.

6.5.1.1 Regularity of Sub-Level solutions

Let $V : \overline{B(r)} \to \mathbb{R}$ is continuous and positive definite and set $S = \left\{x \in \overline{B(r)} : V(x) < c\right\}$. Since $V(0) = 0$, $S_c \neq \emptyset$, for all $c > 0$. Let $\psi_1, \psi_2 : [0, r] \to [0, \infty]$ be continous strictly increasing functions such that $\psi_1(\|x\|) \leq V(x) \leq \psi_2(\|x\|), x \in \overline{B(r)}$. We want to show there are numbers ρ_1, ρ_2 such that $B(\rho_1) \subset S_c \subset B(\rho_2)$. $\rho_1 = \psi_2^{-1}(c)$, $\rho_2 = \psi_1^{-1}(c)$. $x \in B(\rho_1) \iff \|x\| < \rho_1 \implies V(x) \leq \psi_2(\|x\|) < \psi_2(\rho_1) = c \implies x \in S_c$. $x \in S_c \implies \psi_1(\|x\|) \leq V(x) < c \implies \psi_1^{-1}(c) = \rho_2 \implies x \in B(\rho_2)$

6.6 Lyapunov Method

$$x_1' = x_2, x_2' = -x_1 - x_1^2 x_2$$

$(0,0)$ is a critical point. We want to linearize this system at $(0,0)$. We might say $x_1' = x_2, x_2' = -x_1$. The eigenvalues are $\pm i$, so there is no conclusion about the stability of $(0,0)$ in nonlinear problem. Instead, we have the Lyapunov approach. Let $V(x) = x_1^2 + x_2^2$ if ϕ is any solution of the system

$$\frac{d}{dt}V(\phi(t)) = \frac{d}{dt}\left(\phi_1^2 + \phi_2^2\right) = 2\phi_1\phi_1' + 2\phi_2\phi_2' = -2\phi_1^2\phi_2^2 \leq 0$$

$V(\phi(t))$ is monotone decreasing and strictly positive unless $\phi_1 = \phi_2 = 0$ and ${}_{t\to\infty}V(\phi(t)) = V_0$ exists, $V_0 \geq 0$. It also implies $(0,0)$ is a stable equilibrium. In this example V is the Lyapunov function. The key property is that $t \to V(\phi(t))$ is monotone decreasing when ϕ is a solution of the given ODE system. Use the following setup.

Let $f(x_0) = 0$, $u = u(x_0)$ is an open set in $\mathbb{R}^N$ containing x_0. $V : U \to \mathbb{R}$, $V \in C(\mathbb{R})$. $V(x_0) = 0$, $V(x) > 0$, $x \neq x_0$ (positive definiteness condition). $t \to V(\phi(t))$ monotone decreasing if is a solution of $x' = f(x)$. Then we say V is a Lyapunov function of the system $x' = f(x)$.

Example : $V(x) = x_1^2 + x_2^2$ in the above example has all of these properties.

Example: Hamiltonian System. $x \to (p, q)$ $q' = \frac{\partial H}{\partial p}$ $p' =$ for some function $H(p, q)$. The Hamiltonian function

$$\frac{d}{dt}H(p(t), q(t)) = \frac{\partial H}{\partial p}p' + \frac{\partial H}{\partial q}q' = -\frac{\partial H}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} = 0$$

So $H(p(t), q(t))$ is constant on any solution. Suppose p, q are both 1D, and then suppose $H(p, q) = ap^2 + bq^2$. Positive definiteness depends on a, b. Since $H(p, q)$ is constant on a solution, it is a level curves of a hyperbola if $ab < 0$, or ellipse if $ab > 0$. In the latter case, either $\pm H$ is PD and thus is the Lyapunov function, but no such function exists in the former case. So H may or may not be a Lyapunov function depending on details of H and equillibrium point (p_0, q_0).

6.6.0.2 Lie Derivative Assume the system is $x' = f(x)$, $f \in C^1$. $V : \mathbb{R}^N \to \mathbb{R}$ is C^1 also if $x = \phi(t)$ is a solution.

$$\frac{d}{dt}V(\phi(t)) = V(\phi(t)) \cdot \phi'(t) = V(\phi(t)) \cdot f(\phi(t)) = W(\phi(t))$$

where W is continuous and is the derivative of V along a solution. W is called the Lie Derivative of V along the vector field f. Note that we don't need to know any solution $\phi(t)$ to compute W. In the above example, we have $W(x) = -x_1^2 x_2^2$ and $W(x) = 0$ respectively. Notice that $W \leq 0$ in these cases, so we say W is negative semidefinite.

6.6.0.3 Lie Derivative and Stability Theorem: Assume V is a Lyapunov functions. If W is negative sem-definite, then x_0 is stable. If W is negative definite then x_0 is asymptotically stable.
Proof: WLOG let $x_0 = 0$. By previous lemma there exist $r > 0$ and ψ_1, ψ_2 which are continuous on $C[0, r]$ strictly increasing, $\psi_1(0) = \psi_2(0) = 0$ such that $\phi_1(\|x\|\ ) \leq \psi_2(\|x\|\ )$, $\|x\|\ \leq r$. Pick $\epsilon \in (0, r)$, $c = \psi_1(\epsilon)$, $\delta = \psi_2^{-1}(c)$ so $B(\delta) \subset \left\{x \in \overline{B(r)} : V(x) < c\right\} \subset B(\epsilon)$. Let the starting point of a solution $\xi \in B(\delta)$ $x = \phi(t)$ will be the solution of the system with $\phi(0) = \xi$. Then $W \leq 0 \implies V(\phi(t)) < V(\phi(0)) = V(\xi) < c$ for all $t > 0$. This implies $\|\phi(t)\|\ < \epsilon$ for all $t > 0$ so 0 is stable.
Now suppose W is negative definite. Then there exist $\psi_3 \in C[0, r]$ strictly increasing st $W(x) \leq -\psi_3(\|x\|\ )$. $\frac{d}{dt}V(\phi(t)) = W(\phi(t)) < 0$ if $\phi(t) \neq 0$. This implies

$$\psi_1(\|\phi(t)\|\ \leq V(\phi(t)) \leq V(\xi) - \int_0^t \psi_3(\|\phi(s)\|\ )ds$$

by FTC and $W \leq -\psi_3$. If $V(\phi(t))$ does not converge to 0, then it is bounded below say by c_0. We see

$$\psi_2(\|\phi(t)\|\ ) \geq V(\phi(t)) \geq c_0 \implies \|\phi(t)\|\ \geq \psi_2^{-1}(c_0) = \rho_0$$

this implies

$$0 < \psi_1(\rho_0) \leq V(\xi) - \int_0^t \psi_3(\rho_0)ds = V(\xi) - t\psi_3(\rho) \to -\infty, t \to \infty$$

Thus we have reached a contradiction, which implies $V(\phi(t)) \to 0$. $\phi(t) \to 0$ since $\|\phi(t)\|\ \leq \psi_1^{-1}(V(\phi(t)) \to 0$.
Example, $x_1' = -x_1 - x_2$, $x_2' = 2x_1 - x_2^3$. Choose $V(x) = 2x_1^2 + x_2^2$. The properties of $V(x)$ can be checked showing $(0, 0)$ is asymptotically stable.

6.6.0.4 Stability via Lyapunov Functions Theorem: $x = 0$ a rest point of $x' = f(x)$. Let $V : B(0) \to (0, \infty)$ be continuous, positive definite. $W = V \cdot f$ negative semi definite implies $x = 0$ is stable OR if it is negative definite implies $x = 0$ is asymptotically stable.

6.7 Gradient Sytems

$x' = F(x)$ where $F(x) = -f(x)$ for some $f : \mathbb{R}^n \to \mathbb{R}$. Choose $V(x) = f(x)$. Then

$$V \cdot F = f \cdot F(x) = -\|f(x)\|_2$$

Suppose x_0 is an isolated rest point of $x' = F(x) = -f(x)$ and that $f(x)$ has a strict local minimum at x_0. Then x_0 is asymptotically stable rest point of $x' = F(x)$. Suppose x_0 is an isolated rest point of $x' = F(x) = f(x)$ and that $f(x)$ has a strict local minimum at x_0. Then x_0 is an asymptotically stable rest point of $x' = F(x)$. $f(x)$ has a strict local minimum at x_0 implies $f(x) - f(x_0) > 0$. For $x \in \{x : 0 < \|x - x_0\|\ < \delta_1\}$. x_0 an isolated rest point implies $F(x) \neq 0$ in some neighborhood.

6.8 ??

$x' = f(t,x)$, $x(t_0) = x_0$ with solution $\phi(t,t_0,x_0)$.
Let $x = \psi(t)$ be a solution defined for $t_0 \leq t \leq \infty$ we say $\psi(t)$ is stable (Lyapunov) if ther eis a $b > 0$ such that $\|x_0 - \psi(t_0)\| \leq b$ implies $\phi(t,t_0,x_0)$ exists for all $t \geq t_0$, and given $\epsilon > 0$ there is a $\delta = \delta(\epsilon, \psi(t_0), f) \in (0,\beta)$ such that $\|x_0 - \psi(t_0) - \psi(t)\| < \epsilon$ for all $t \geq t_0$. $\psi(t)$ is asymptotic stable if in addition there is another $\delta \in (0,b)$ such that $\|x_0 - \psi(t_0)\| < \delta$ implies $\|\phi(t,t_0,x_0) - \psi(t)\| \to 0$ as $t \to \infty$. Suppose $x = \psi(t)$ is a solution for $t \geq t_0$. Set $y = x - \psi(t)$. Then

$$y' = x' - \psi'(t) = f(t,x) - f(t,\psi(t)) = A(t)y + h(t,y)$$

where $A(t) = f_x(t,\psi(t))$, $h(t,y) = f(t, y+\psi(t)) - f(t,\psi(t)) - f_x(t,\psi(t))y$.

$$h(t,y_1) - h(t,y_2) = \int_0^1 \left(f_x(t, y_2 + s(y_1 - y_2) + \psi(t)) - f_x(t,\psi(t))\right)(y_2 - y_1)ds$$

So $0 \leq s \leq 1$ implies

$$\|(y_2 + s(y_1 - y_2) + \psi(t)) - \psi(t)\| = \|sy_1 + (1-s)y_2\| \leq s\|y_1\| + (1-s)\|y_2\|$$

If $\|y_1\|, \|y_2\| < \delta$, then $\|(y_2 + s(y_1 - y_2)\psi(t)) - \psi(t)\| \leq \delta$. So $f_x(t,y)$ is uniformly continous as long as we restrict t to lie in $[t_0, t_0 + T]$ for some $T > 0$ (and assume $\|y_2\|, \|y_1\| \leq \rho$). This implies for all $\epsilon > 0$, $\delta > 0$, such that $\|h(t,y_1) - h(t,y_2)\| < \epsilon$ provided $t \in [t_0, t_0+T]$ and $y_1, y_2 \in B_\delta$.
Now suppose $\psi(t)$ is a $T-$periodic solution of $x' = f(t,x)$ where $f(t+T,x)$. $y = x - \psi(t)$ implies $y' = A(t)y + h(t,y)$. $A(t) = f_x(t,\psi(t))$ so $A(t+T) = f_x(t+T,\psi(t)) = f_x(t,\psi(t)) = A(t)$.

$$h(t+T,y) = f(t+T,y) - f(t+T,\psi(t+T)) - f_x(t+T,\psi(t+T))y = h(t,y)$$

Theorem: If the Floquet Multipliers of $x' = A(t)x$ all lie in $\{z \in : |z| < 1\}$, then $\psi(t)$ is asymptotically stable.
Idea of Proof: Let $\Phi(t) = P(t)e^{tR}$ be the fundamental matrix with $\Phi(0) = I$. We have $P(0) = I$ and $P(t+T) = P(t)$, $t \in \mathbb{R}$. If λ is a Floquet multiplier then $\lambda = e^{T\rho}$, where ρ is an eigenvalue of R. $|\lambda| < 1 \iff Re(\rho) < 0$.

$$P^{-1}(t)\left(y(t) = P(t)e^{tR}y(0) + P(t)e^{tR}\int_0^t e^{-sR}P(s)h(s,y(s))ds\right)$$

gives us for $w(t) = P^{-1}(t)y(t)$,

$$w(t) = e^{tR}w(0) + \int_0^t e^{(t-s)R}P^{-1}(s)h(s,P(s)w(s))ds$$

which implies

$$w' = Rw + g(t,w), \qquad g(t,w) = P^{-1}h(t,P(t)w(t))$$

and we note that $w = 0 \iff y = 0$. Note

$$\|g(t,w_1) - g(t,w_2)\| \leq \left\|P^{-1}(t)\right\| \|P(t)w_1 - P(t)w_2\| \epsilon_1 \leq \epsilon_1 \left\|P^{-1}(t)\right\|_\infty \|P(t)\|_\infty \|w_1 - w_2\| \leq \epsilon\|w_1 - w_2\|$$

Given that we can choose $\epsilon_1 < \epsilon(\left\|P^{-1}(t)\right\|_\infty \|P(t)\|_\infty)^{-1}$
Now we use a slightly modified proof of asymptotic stability for $w = 0$ solution of $w' = Rw + g(t,w)$ to show $y = 0$ is asymptotic stable solution of $y' = A(t)y + h(t,y)$ and $\psi(t)$ is an asym. stable solution of $x' = f(t,x)$.
Note. Suppose $x = \psi(t)$ is a $T-$periodic solution of $x' = f(x)$. It turns out that the linear variation system in this case always has a Floquet multiplier $\lambda = 1$. This is seen as follows.

$$\psi'(t) = f(\psi(t)) \implies \psi''(t) = f_x(\psi(t))\psi'(t) \implies (\psi')'(t) = A(t)\psi'(t)$$

which implies $\psi'(t)$ is a solution of $y' = A(t)y$. So $\psi'(t) = \Phi(t)\psi'(0)$ implies $\psi'(T) = \Phi(T)\psi'(0)$. $\psi(t+T) = \psi(t)$ implies $\psi'(T) = \psi'(0)$ implies $\Phi(T)\psi'(0) = \lambda\psi'(0) = (1)\psi'(0)$

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6.9 Invariant Sets

$$x' = f(x) \qquad f \in C^1(\Omega), \Omega \in \mathbb{R}^N, open$$

Definition: A set of points $E \in \Omega$ is said to be a positively /negatively (respectively) invariant if for each $x_0 \in E$ the solution $\phi(t, x_0)$ of the above equation through satisfies $\phi(t, x_0) \in E$ for all $t \geq 0$, $t \leq 0$ (respectively). Sets are invariant if they are both positvely and negatively invariant. Examples:

- If $f(x_0) = 0$ then $\{x_0\}$ is an invariant set
- If $\phi(t, x_0)$ is $T-$ periodic thne the solution will be the orbit

$$(x_0) = \{\phi(t, x_0) : 0 \leq t \leq T\}$$

- Consider $x' = f(x) = -x(1 - x^2)$. $\{(-.5, .5)\}$ is positively invariant, $\{(.5, 1.5)\}$ is negatively invariant. $\{(-1, 1)\}$ is invariant. $[(-1, 1)]$ is invariant.
- If $V \in C^1(\Omega)$ is a Lyapunov function for the above equation, $f(0) = 0$, and $V(x) \cdot f(x) \leq 0$, $x \in B_\delta$ for some $\delta > 0$ then $S_c = \{x \in B_\delta : V(x) \leq C\}$ is positively invariant for $c > 0$.

Definition. Suppose $\phi(t, x_0)$ is a solution that exists for all $t \geq 0$. The positive limit set, denoted by $\omega(x_0)$, of x_0 (or $\phi(t, x_0)$) is the set of all points $y \in \Omega$ for which there is a sequence of times $\{t_n\}$ satisfying (1) $t_n \to \infty$ as $n \to \infty$ and (2) $\phi(t_n, x_0) \to y$ as $n \to \infty$. The α-limit set is $\alpha(x_0) = \cap_{\tau \leq 0} \{\phi(t, x_0) : t \leq \tau\}$.
Lemma: If the oslution $\phi(t, x_0)$ exusts for all $t \geq 0$ and the orbit $^+(x_0) = \{\phi(t, x_0) : t \geq 0\}$ remains in a compact set $K \subset \Omega$, then $\omega(x_0)$ is a nonempty compact subset of Ω that is invaraint. Furthermore $\text{dist}(\phi(t, x_0), \omega(w_0)) \to 0$ as $t \to \infty$ (although not uniquely).
Proof. $\omega(x_0)$ is nonempty: Let $\{t_n\} \subset [0, \infty)$ be a sequence with $t_n \to \infty$. Then $\{\phi(t_n, x_0)\}$ is a sequence contained in K. Hence there is a subsequence $\{\phi(t_{n_k}, x_0)\} \subset K$ and a point $y \in K$ such that $\phi(t_{n_k}, x_0) \to y$ as $h \to \infty$. This implies $y \in \omega(x_0) \neq \emptyset$.
$\omega(x_0)$ is closed: Suppose $\{x_n\} \subset \omega(x_0)$ and $x_n \to y$ as $n \to \infty$. For each n there is a $t_n > n$ such that

$$\|\phi(t_n, x_0) - x_n\| < \frac{1}{n}$$

which implies

$$\|\phi(t_n, x_0) - y\| \leq \|\phi(t_n, x_0) - x_n\| + \|x_n - y\| < \frac{1}{n} + \|x_n - y\|$$

So $\phi(t_n, x_0) \to y$ as $n \to \infty$ so $y \in \Omega(w_0)$. This means omega limit set is compact since is a closed subset of a compact set K.
$\omega(x_0)$ is invariant: Let $y \in \omega(x_0)$ and choose $\{t_n\}$ so that $x_n = \phi(t_n, x_0) \to y$ as $n \to \infty$. By continuous dependence we know $\phi(t, x_n) \to \phi(t, y)$ as $n \to \infty$, for each $t \in (\alpha(y), \beta(y))$ where $(\alpha(y), \beta(y))$ is the maximal interval of existence of $\phi(t, y)$. But then $\phi(t + t_n, x_0) = \phi(t, \phi(t_n, x_0)) = \phi(t, x_n) \to \phi(t, y)$ as $n \to \infty$. This implies $\phi(t, y) \in \omega(x_0)$ for all $t \in (\alpha(y), \beta(y))$. $\omega(x_0) \subset K \implies (\alpha(y), \beta(y)) = (-\infty, \infty)$. $\phi(t, x_0) \to \omega(x_0)$ as $t \to \infty$. We need to show that for any $\epsilon > 0$ there is a $T \geq 0$ such that $\text{dist}(\phi(t, x_0), \omega(w_0)) < \epsilon$. Suppose this is not true. Then for some $\epsilon > 0$ there is a sequence $\{t_n\}$ such that $\text{dist}(\phi(t_n, x_0), \omega(w_0)) \geq \epsilon$. But $\{\phi(t_n, x_0)\} K$ implies there is a subsequence $\{\phi(t_n, x_0)\}$ and a point $y \in K \phi(t_{n_k}, x_0) \to y$ as $k \to \infty$. So $t \in \omega(x_0)$. Therefore there are times t_n such that $\text{dist}(\phi(t, x_0), \omega(w_0)) < \epsilon$.

6.9.0.5 Lemma

$$x' = f(x) \qquad x(0) = x_0$$

$\phi(t, x_0)$ exists for all $t \geq 0$. $^+(x_0) = \{x = \phi(t, x_0) : t \geq 0\} \subset K$, compact which implies $\omega(x_0)$ non-empty, compact, invariant and $\phi(t, x_0) \to \omega(x_0)$ as $t \to \infty$. Also $\omega(x_0)$ is also connected. Ie $\omega(x_0)$ cannot be decomposed into 2 disjoint closed sets.
Proof: If not e have $\omega(x_0) = U \cup V$ where U, V are closed (therefore compact) and $U \cap V = \emptyset$. Let

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$d = \text{dist}(U, V)$. Since U, V are compact and disjoint, $d > 0$. But then there are sequences of times $\{t_{2n}\}$ and $\{t_{2n+1}\}$ such that $t_{n+1} > t_n$ for all n and $\text{dist}(\phi(t_{2n}), U) < \frac{1}{3}d$, $\text{dist}(\phi(t_{2n+1}), V) < \frac{1}{3}d$. Since the distance function is continuous, the function $f(t) = \text{dist}(\phi(t, x_0), U)$ is continuous. We have $f(t_{2n}) < \frac{d}{3}$ and $f(t_{2n+1}) > \frac{2d}{3}$. So by IVT, there is a time $\tilde{t}_n$, $t_{2n} < \tilde{t}_n < t_{2n+1}$ such that $\text{dist}(\phi(\tilde{t}_n, x_0), U) > \frac{1}{3}d$ and $\text{dist}(\phi(\tilde{t}_n, x_0), V) > \frac{1}{3}d$. Hence $\tilde{n} \to \infty$ and $\phi(\tilde{t}_n, x_0) \to \omega(x_0)$ and $\omega(x_0) \notin U \cup V$: a contradiction.

6.9.0.6 Lemma

$$x' = f(x) \qquad f \in C^1(\Omega)$$

Let $V \subset C^1(\Omega)$ and $W(x) = V(x) \cdot f(x) \leq 0, x \in \Omega$. Suppose $x_0 \in \Omega$ and $\phi(t, x_0)$ exists for all $t \geq 0$with $^+(x_0) \subset K$,a compact set, so that $\omega(x_0)$ is non-empty. Then $W(x) = 0, x \in \omega(x_0)$.
Proof: We have $\frac{d}{dt}V(\phi(t, x_0)) = (\phi(t, x_0)) \cdot f(\phi(t, x_0)) \leq 0$ which implies $t \to (\phi(t, x_0))$ is a non-increasing function. Also $V(\phi(t, x_0))$ is bounded below since $\phi(t, x_0) \in K$, $t \geq 0$. Then $\lim_{t\to\infty} V(\phi(t, x_0)) = V_\infty$, which is some number. Now suppose $y_1, y_2 \in \omega(x_0)$. There are sequences $\{t_n\}, \{s_n\}$ such that $\phi(t_n, x_0) \to y_1$ and $\phi(s_n, x_0) \to y_2$ where $t_n, s_n \to \infty$. Then the continuity of V implies

$$\lim_{n\to\infty} V(\phi(t_n, x_0)) = V(y_1) = V_\infty = V(y_2) = \lim_{n\to\infty} V(\phi(s_n, x_0))$$

Therefore $V(x)$ is constant on $\omega(x_0)$ which implies $W(x) = 0$, $x \in \omega(x_0)$.

6.9.0.7 Theorem

$$x' = f(x) \qquad f(0) = 0$$

Suppose $V \in C^1(B_r)$ is positive definite, $W(x) = V \cdot f(x) \leq 0$, $x \in B_r$ and $S_c = \{x \in B_r : V(x) \leq c\}$ is contained in a compact subset of B_r for some $c > 0$. If the only invariant subset of the set $z = \{x \in B_r : W(x) = 0\}$ is $\{0\}$, then $x = 0$ is asymptotically stable.
Proof: Choose $x_0 \in S_c$. Since $V(\phi(t, x_0)) \leq c$, $t \geq 0$. We have $^+(x_0)$ remains in a compact set. Hence $\omega(x_0)$ is nonempty and invariant. Also $\omega(x_0) \in z$, but then $\omega(x_0) = \{0\}$ for any $x_0 S_c$.

6.9.0.8 Example

Consider

$$x' = (A - By)x \qquad y' = (Cx - D)y$$

The rest points are $x = 0$ or $y = \frac{A}{B}$. $y = 0$ or $x = \frac{D}{C}$. The invariant sets are $L_1 = \{(x, y) : x = 0, y \geq 0\}$, $L_2 = \{(x, y) : x \geq 0, y = 0\}$, $Q = \{(x, y) : x > 0, y > 0\}$. We can remove the dimensions in space by the transformation $\tilde{x}\frac{D}{C}$, $\tilde{y}\frac{A}{B} = y$. We can scale time by using $\tau = At$ and let $a = \frac{D}{A}$. So our new equations are

$$\tilde{x}' = (1 - y)x \qquad \tilde{y}' = a(x - 1)y$$

Set $f(x) = x - \ln(x) - 1$. Consider $V(x, y) = af(x) + f(y)$. This is constant along solutions since its an integral curve of the separable equation. So by construction, $V(1, 1) = 0$, $V(x, y) > 0$.

6.9.0.9 Example

Consider

$$x' = (1 - y - \lambda x)x = \alpha(x, y)x \qquad y' = a(x - 1 - \mu y)y = \beta(x, y)y$$

Where $a, \lambda, \mu > 0$. There are four rest points. The one in the first quadrant excluding the axes is

$$(x_0, y_0) = \left(\frac{1 + \mu}{1 + \lambda\mu}, \frac{1 - \lambda}{1 + \lambda\mu}\right)$$

To create a Lyapunov function for this critical point, we consider a perturbed version of the function in the previous problem.

$$V(x, y) = \gamma_1 f\left(\frac{y}{y_0}\right) + \gamma_2 a f\left(\frac{x}{x_0}\right)$$

We seek to determine γ_1, γ_2. We see

$$\frac{\partial V}{\partial x} = \frac{\gamma_2}{x_0} a f'\left(\frac{x}{x_0}\right) = \gamma_2 a \frac{x - x_0}{x x_0} \qquad \frac{\partial V}{\partial y} = \frac{\gamma_1}{y_0} a f'\left(\frac{y}{y_0}\right) = \gamma_1 \frac{y - y_0}{y y_0}$$

For convenience write

$$\alpha(x,y) = \alpha(x,y) - \alpha(x_0,y_0) = -(y-y_0) - \lambda(x-x_0) \qquad \beta(x,y) = \beta(x,y) - \beta(x_0,y_0) = a((x-x_0) - \mu(y-y_0))$$

So then

$$\left\langle \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y} \right\rangle \cdot \langle x\alpha(x,y), y\beta(x,y)\rangle = a\left(\frac{-\gamma_2}{x_0}\left((x-x_0)(y-y_0) + \lambda(x-x_0)^2\right) + \frac{\gamma_1}{y_0}\left((y-y_0)(x-x_0) - \mu(y-y_0)\right)\right)$$

So if we choose $\gamma_2 = x_0$, $\gamma_1 = y_0$,

$$\left\langle \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y} \right\rangle \cdot \langle x\alpha(x,y), y\beta(x,y)\rangle = -a\left(\lambda(x-x_0)^2 + \mu(y-y_0)^2\right)$$

So then

$$V(x,y) = y_0 f\left(\frac{y}{y_0}\right) + a x_0 f\left(\frac{x}{x_0}\right)$$

We see that $V(x_0,y_0) = 0$, $V(x,y) > 0$ for $(x,y) \neq (x_0,y_0)$, and $V \cdot \langle x\alpha(x,y), y\beta(x,y)\rangle$ is negative definite so we conclude that (x_0,y_0) is an asymptotically stable rest point.